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A Study of Characterization of Some Regular Gamma Rings

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University of Rajshahi

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A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS , UNIVERSITY OF RAJSHAHI FOR THE DEGREE

OF

MASTER OF PHILOSOPHY

SUBMITTED

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It is certified that the thesis entitled " **A Study of Characterization of Some Regular Gamma Rings** " submitted by Ayesha Nazneen contains the fulfillment of all the requirements for the degree of Master of Philosophy in Mathematics under the University of Rajshahi , has been completed under my supervision . I do believe that this research work is an original one , and it has not been submitted elsewhere for any degree .

Akhil ch. Paul
Professor Akhil Chandra Paul

Supervisor

STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief does not contain any material previously published or written by another person except where due reference is made in the text.

Ayesha Nazneen

[Ayesha Nazneen]

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ABSTRACT

The present thesis entitled , "A Study of Characterization of Some Regular Gamma Rings " is the outcome of researches carried out by me under the close supervision of Dr . Akhil Chandra Paul , Professor , Department of Mathematics , Rajshahi University . The thesis is of six chapters . In the first chapter we have tried to introduce all types of the conceptions of the complete thesis .

In the second chapter we have given the definition of Γ - ring due to Barnes and of the relevant things . Various types of Γ- rings and their examples are also presented there . Some kinds of radical and corresponding theorems are also stated and important ones are proved .

The definition of k - regular Γ- ring is given in the third chapter . Kyuno defined this regular Γ- ring . We have tried to prove that the class of all k - regular Γ- rings forms a radical . Some of the characterizations of this Γ- rings are developed . We have also shown that k - regular Γ- ring without zero divisors is a skew Γ- field .

In the fourth chapter we have studied the von Neumann regular Γ - rings . We have developed some properties of this Γ- rings . We have also shown that the class of all von Neumann regular Γ- rings is a radical .

We have generalized Jacobson Γ- rings and Jacobson radical in the fifth chapter . The primitive Γ- rings and the special class of Γ - rings have also been placed there . We have proved that the Jacobson radical for Γ- rings is a special class of radicals .

In the sixth chapter we have defined a radical determined by the maximal ideals of a gamma ring . We have studied some of the properties of this radical .

We have included a complete bibliography of the works which have been used as my very helpful references to finish my thesis .

At the end of this thesis we have given a list of important symbols which are used in this thesis .

CONTENTS

CHAPTER – ONE

Introduction

The concepts of a Γ- ring was first introduced by Nobusawa [29] in 1964. His concept is more general than a ring . He obtained analogues of the Wedderburn Theorems for Γ- rings with minimum condition on left ideals. Now a day , his Γ- ring is called a Γ- ring in the sense of Nobusawa .

W. E . Barnes [5] gave a definition of a Γ- ring which is more general. He introduced the notion of Γ- homomorphism , prime and primary ideals m - systems and the radical of an ideal for Γ- rings . He also gave the definition of residue class Γ- rings .

Coppage and Luh [11] introduced the notion of Jacobson radical , Livitzki nil radical , nil radical and strongly nilpotent radical for Γ- rings , and obtained some basic radical properties for a Γ- ring .

Shoji Kyuno , Nobusawa and B-smith [21] studied regular Γ- rings. They developed various properties of regular gamma rings . They also showed that the class of all regular Γ- rings is a radical .

G . L . Booth , N . J . Groenewald and W. A . Oliver [9] defined a general regularity for Γ- rings and to explore ways of generating such regularities . They showed that regularities for Γ- rings can be generated by means of polynomials .They also introduced a radical which lies between the Jacobson radical and the right Brown – McCoy radical .

A Brown – McCoy radical for Γ- rings was introduced by G. L. Booth [7] . He defined G – regularity for a Γ- ring and then Brown – McCoy radical is defined . He obtained some properties of this radical .

A weak form of von Neumann regularity for Γ- rings was introduced by Tan in [32] . An element of a Γ- ring M is called F- regular by Tan if $a \in \langle a \rangle_M \Gamma M \Gamma \langle a \rangle_M$. A Γ- ring M is called F- regular if each element of M is F- regular . He showed that the class of all F- regular Γ- rings is a radical .

The concept of von Neumann regularity for Γ- rings was introduced by Chen in [10] . He showed that $R(M) = \{a \in M : \langle a \rangle_M$ is von Neumann regular in $M\}$ is the greatest von Neumann regular ideal of M . He also obtained the result , " the class $\{ M : R(M) = M , M$ a Γ- ring $\}$ forms a radical ".

In the first chapter, an introduction of the previous works relevant to regularities and radicals of Γ- rings are given .

In the second chapter we have defined Γ- ring due to Barnes . Some examples of Γ- rings are given . We have discussed the definition of some necessary topics such as ideal of a Γ- ring , right operator ring , nilpotent Γ- ring , quasi – regular Γ- rings , Γ- homomorphism , radical , hereditary radical , residue class Γ- rings , etc . Some corresponding theorems are also placed in this chapter and some of them are proved .

In the third chapter we have defined k - regular Γ- rings . We have shown that residue class Γ- ring of a k - regular Γ- ring is also a k - regular Γ- ring .We studied some properties of this regular Γ- rings . We have also shown that the class of all k - regular Γ- rings is a radical .

In the fourth chapter we have defined von Neumann regular Γ- rings . Various properties of this Γ- rings have been studied . We have shown that the class of all von Neumann regular Γ- rings is a radical .

Jacobson Γ- rings and Jacobson radicals have been characterized in the fifth chapter . We have also defined primitive Γ- rings and special class of Γ- rings . We have proved in this chapter that the Jacobson radical is the largest radical for which primitive Γ- rings are semi simple . Also we have proved that the Jacobson radical is a special radical .

In the sixth chapter we have studied the radical determined by the maximal ideals of a Γ- ring . We have characterized this type of radicals by means of the regularity properties of Γ- rings .

A bibliography is given at the end of my thesis .

CHAPTER - TWO

Gamma rings and its preliminaries

In this chapter we have discussed the concepts of gamma rings, ideal of a Γ- ring, right and left operator ring, nilpotent Γ- ring, nil Γ- ring, quasi - regular Γ- ring etc. Here we have also given the definitions of a radical, Jacobson radical etc. Some theorems relevant to these concepts are also given.

2 Definition.

2.1 Gamma ring : Let M and Γ be two abelian groups. Suppose that there is a mapping (composition) from $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x \alpha y$) such that

(i) $(x+y)\alpha z = x\alpha z + y\alpha z$,

$x(\alpha+\beta)z = x\alpha z + x\beta z$,

$x\alpha(y+z) = x\alpha y + x\alpha z$,

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a **Γ- ring.**

If the conditions of the above definition are strengthened to

1. $x\alpha y$ is an element of M, $\alpha x \beta$ is an element of Γ,
2. same as (i)
3. $(x\alpha y)\beta z = x(\alpha y \beta)z = x\alpha(y\beta z)$
4. $x\alpha y = 0$ for all x, y in M implies $\alpha = 0$,

then M is called a Γ- ring in the sense of Nobusawa.

2.2 Examples : Let X and Y be abelian groups . Let $M = \mathrm{Hom}(X,Y)$ and $\Gamma = \mathrm{Hom}(Y, X)$ and $x \alpha y$ the usual composite map for all $x, y \in M$ and $\alpha \in \Gamma$. Then clearly (i) and (ii) conditions are satisfied and M is a Γ-ring .

Every ring M is a Γ-ring if we take $\Gamma = M$ and interpret the above operation in the natural ways .

2.3 Ideal of Γ- rings : A subset A of the Γ-ring M is a right (left) **ideal** of M , if A is an additive subgroup of M and $A \Gamma M = \{ a \alpha c : a \in A, \alpha \in \Gamma, c \in M \}$ $(M \Gamma A)$ is contained in A .

If A is both a right and a left ideal , then we say that A is an ideal , or two sided ideal of M .

If A and B are both right (respectively left or two sided) ideals of M , then $A + B = \{a + b \mid a \in A , b \in B \}$ is clearly a right (respectively left or two sided) ideal , called the sum of A and B . We can say every finite sum of right (respectively left or two sided) ideal of a Γ-ring is also a right (respectively left or two sided) ideal .

It is clear that the intersection of any number of right (respectively left or two sided) ideals of M is again a right (respectively left or two sided) ideal of M .

If A is a left ideal of M , B is a right ideal of M and S is any non empty subset of M , then the set

$A\Gamma S = \{\sum a_i \gamma_i s_i \mid a_i \in A, \gamma_i \in \Gamma, s_i \in S$, n is any positive integer$\}$ is a left ideal of M and $S\Gamma B$ is a right ideal of M. $A\Gamma B$ is a two sided ideal of M.

If $a \in M$, then the **principal ideal generated by a** denoted by $< a >$ is the intersection of all ideals containing a and is the set of all finite sums of elements of the form $n a + x \alpha a + a \beta y + u \gamma a \delta v$, where n is an integer, x, y, u and v are elements of M and $\alpha, \beta, \gamma, \delta$ are elements of Γ. This is the smallest ideal generated by a.

Let $a \in M$. The smallest right (left) ideal generated by a is called the principal right (left) ideal and is denoted by $|a>$ ($<a|$).

2.4 Semiprime : An ideal P of a Γ-ring M is said to be **semiprime** if for any ideal Q, $Q \Gamma Q \subseteq P$ implies $Q \subseteq P$. A Γ-ring M is semiprime if the zero ideal is semiprime.

If A is a semiprime ideal and B is an ideal, $B \subseteq A$, then $(B\Gamma)^n = (B \Gamma B \Gamma B \Gamma \ldots\ldots\ldots B\Gamma) B \subseteq A$ for an arbitrary positive integer n.

Now we state some theorems relevant to these definitions.

2.5 Theorem : If Q is an ideal of Γ-ring M, then the following conditions are equivalent :

(i) Q is semiprime

(ii) if $a \in M$ such that $<a>\Gamma<a> \subseteq Q$, then $a \in Q$.

Proof : Let Q be a semiprime ideal and for any a in M we have that $<a>\Gamma<a> \subseteq Q$, then by the definition of semiprime ideal, $<a> \subseteq Q$, therefore, $a \in Q$.

Again, suppose that U is an ideal of M and let $U\Gamma U \subseteq Q$. Then for any a in U, $<a>\Gamma<a> \subseteq Q$ implies $a \in Q$. Therefore, $U \subseteq Q$, thus Q is semiprime.

2.6 Theorem : An ideal Q in a Γ - ring M is a semiprime ideal in M if and only if $P(Q)=Q$, where $P(Q)$ represents the intersection of all prime ideals of M.

2.7 Corollary : If Q is an ideal in a Γ - ring M, then $P(Q)$ is the smallest semiprime ideal in M which contains Q.

2.8 Theorem : $P(M)$ is the semiprime ideal which is contained in every semi prime ideal in M.

2.9 Theorem : If Q is an ideal in a Γ- ring M, then the following conditions are equivalent.

(i) Q is semiprime ideal

(ii) if $a \in M$ such that $a\Gamma M\Gamma a \subseteq Q$, then $a \in Q$.

(iii) If $<a>$ is a principal ideal in M such that $<a>\Gamma<a> \subseteq Q$, then $a \in Q$.

(iv) if U is a right ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$.

(v) if V is a left ideal in M such that $V\Gamma V \subseteq Q$, then $V \subseteq Q$.

2.10 Corollary : A Γ- ring M is semiprime if and only if $a\Gamma a\Gamma a=0$ implies $a=0$.

2.11 Prime ideal : An ideal P of the Γ-ring M is said to be **prime** if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

2.12 Theorem : An ideal P of M is prime if and only if $<a>\Gamma<b> \subseteq P \Rightarrow a \in P$ or $b \in P$.

Proof : Let P be the prime ideal of M, then we have from the definition $<a> \subseteq P$ or $<b> \subseteq P$. That is $a \in P$ or $b \in P$.

Conversely, suppose $<a>\Gamma<b> \subseteq P$ implies $a \in P$ or $b \in P$. Now let A and B be ideals such that $A\Gamma B \subseteq P$. Now if $A \not\subset P$, then there exists $a_1 \in A$ such that $a_1 \notin P$. Now for any $b_1 \in B$, $<a_1>\Gamma<b_1> \subseteq A\Gamma B \subseteq P$, implies $b_1 \in P$ because $a_1 \notin P$. Therefore, $B \subseteq P$. And so P is prime.

2.13 Theorem : If A and P are ideals of M, $A \subseteq P$ and P is prime, then P/A is prime in M/A. Conversely, if P' is a prime ideal of M/A, f the canonical homomorphism of M onto M/A, then $f^{-1}(P') = P$ is a prime ideal of M.

Proof : First suppose that $A \subseteq P$, where P is prime and $(B/A)\Gamma(C/A) \subseteq P/A$. Then $B\Gamma C \subseteq A$. So that $B\Gamma C \subseteq P$. Since P is prime either $B \subseteq P$ or $C \subseteq P$. That is $B/A \subseteq P/A$ or $C/A \subseteq P/A$, and therefore, P/A is prime.

Conversely, suppose P' is a prime ideal of M/A, f is the canonical homomorphism of M onto M/A. Let $B\Gamma C \subseteq P = f^{-1}(P')$. Then $(f(B))\Gamma(f(C))$

$\subseteq f(P) = P'$ and since P' is prime $f(B) \subseteq P'$ or $f(C) \subseteq P'$. That is $B \subseteq f^{-1}(P')$ or $C \subseteq f^{-1}(P')$. It means that $B \subseteq P$ or $C \subseteq P$. So that P is prime.

2.14 Prime Γ- ring : A Γ- ring M is said to be **prime** if the zero ideal is prime.

2.15 Theorem : If M is a Γ- ring, then the following conditions are equivalent :

(i) M is prime Γ- ring,

(ii) $a, b \in M$ and $a\Gamma M \Gamma b = (0)$ implies $a = 0$ or $b = 0$,

(iii) if $<a>$ and $<b>$ are principal ideals in M such that $<a> \Gamma <b> = (0)$, then $a = 0$ or $b = 0$,

(iv) if A and B are right ideals of M such that $A \Gamma B = (0)$, then $A = (0)$ or $B = (0)$,

(v) if A and B are left ideals in M such that $A \Gamma B = (0)$ then $A = (0)$ or $B = (0)$.

2.16 Theorem : If P is an ideal in a Γ- ring M, then the residue class Γ- ring M/P is a prime Γ- ring if and only if P is a prime ideal in M.

2.17 Lemma : If P is prime ideal of M then $P \cap I$ is a prime ideal of I.

Proof : Let A, B be two ideals of I such that $A \Gamma B \subseteq P \cap I$. If $<A> = A + A\Gamma M + M \Gamma A + M \Gamma A \Gamma M$ and $<B> = B + B \Gamma M + M \Gamma B + M \Gamma B \Gamma M$, then $I \Gamma <A> \Gamma I \subseteq A$ and $<A> \subseteq I$ implies $A \subseteq I$. Similarly we can show that $B \subseteq I$. Now $(<A> \Gamma <A> \Gamma <A>) \; \Gamma \; (<B> \Gamma <B> \Gamma <B>) \subseteq A \Gamma B \subseteq P \cap I \subseteq P$. Since P is prime in M and $<A> \Gamma <A> \Gamma <A>$ and $<B> \Gamma <B> \Gamma <B>$ are ideals of M, We have either $\langle A\rangle\Gamma\langle A\rangle\Gamma\langle A\rangle \subseteq P$ or $\langle B\rangle\Gamma\langle B\rangle\Gamma\langle B\rangle \subseteq P$. This implies that $\langle A\rangle \subseteq P$ or $\langle B\rangle \subseteq P$. Therefore, either $A \subseteq P\cap I$ or $B \subseteq P\cap I$. And so $P\cap I$ is prime.

2.18 Right operator ring : Let M be a Γ-ring and F be the free abelian group generated by $\Gamma \times M$. Then $A = \{\sum n_i(\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum n_i\, a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Then $[\alpha, x] + [\alpha, y] = [\alpha, x+y]$ and $[\alpha, x] + [\beta, x] = [\alpha+\beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$.

We define a multiplication in R by $(\Sigma[\alpha_i, x_i])(\Sigma[\beta_j, y_j]) = \Sigma[\alpha_i, x_i\beta_j y_j]$. Then R forms a ring. Now we define a composition on $M \times R$ into M by $a(\Sigma[\alpha_i, x_i]) = \Sigma a\alpha_i x_i$ for $a \in M$, $\Sigma[\alpha_i, x_i] \in R$, then M is a right R-module, and we call R **the right operator ring of the Γ- ring M**. Similarly we can define L the left operator ring of the Γ-ring M.

2.19 Nilpotent Γ- ring : An element a of a Γ-ring M is nilpotent if for any $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(a\gamma)^n a = (a\gamma)(a\gamma)(a\gamma)(a\gamma)\ldots\ldots(a\gamma)a = 0$. A Γ-ring M is said to be nil if every element of M is nilpotent .

2.20 Quasi - regular Γ- ring : An element a of a Γ-ring M is said to be right quasi - regular (abbreviated rq r) if, for any $\gamma \in \Gamma$, there exist $\delta_i \in \Gamma$, $x_i \in M$, $i = 1, 2, 3, \ldots\ldots, n$ such that $x\gamma a + \Sigma x\delta_i x_i - \Sigma x\gamma a\delta_i x_i = 0$ for all $x \in M$. A Γ-ring M is called right - quasi regular if every elements of M are right - quasi regular.

2.21 Lemma : An element x of a Γ- ring M is rq r if and only if, for all $\gamma \in \Gamma$, $[\gamma, x]$ is rqr in the right operator ring R of M.

2.22 Theorem : Every nilpotent element in a Γ- ring M is rq r.

Proof : Let $a \in M$ be nilpotent. Then for any $\gamma \in \Gamma$ we have $(a\gamma)^n a = 0$ for some n. Let $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \ldots\ldots = \delta_n = \gamma$ and let $x_1 = -a$, $x_i = -(a\gamma)^{i-1}$ for $i = 1, 2, 3, \ldots\ldots, n$. Then $x\gamma a + \Sigma x \delta_i x_i - \Sigma x \gamma a \delta_i x_i = x\gamma(a\gamma)^n a = 0$ for each $x \in M$. Hence a is rqr. Therefore, the proof is completed.

2.23 Γ- homomorphism : Let M and N both be Γ- rings and f a map of M into N. Then f is a Γ- homomorphism if and only if $f(x+y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

If f is one - to - one and onto, then f is a Γ - isomorphism.

If f is a Γ- homomorphism of M into N, then kernel of f i.e.,

$f^{1}(0) = \{ x \in M : f(x) = 0 \}$, which is also an ideal of M. More generally, if B is a right (left, two sided) ideal of N, then $f^{-1}(B) = \{x \in M : f(x) \in B\}$ is also a right (resp. left or two sided) ideal of M. Similarly, if f is a Γ - homomorphism of M onto N and A is any right (left, two sided) ideal of M, then $f(A) = \{ f(a) : a \in A\}$ is a right (left, two sided) ideal of N.

2.24 Theorem : Let I be an ideal of a Γ- ring M and f be the natural mapping $x \to x + I$ of M into M / I .Then f is a Γ- homomorphism of M into M / I with kernel I. Conversely, f is a Γ- homomorphism of M into a Γ- ring N and I is the kernel of f, then M / I is Γ- isomorphic to N.

The Γ- homomorphism defined in this theorem is called natural homomorphism which is also defined as the following way.

Let A be an ideal of a Γ- ring M. Then the ordered pair (f, I) of mappings, where, $f: M \to M/A$ is defined by $f(x) = x + A$, and I is the identity mapping of Γ, is a homomorphism called the natural homomorphism from M onto M/A.

2.25 Theorem : Let f be a Γ- homomorphism of a Γ- ring M into a Γ- ring N with kernel I. Then J' is an ideal of N if and only if $f^{-1}(J') = J$ is an ideal of M containing I. In this case we have M/J, N/J' and $(M/I)/(J/I)$ are all Γ- homomorphism.

2.26 Theorem : Let I and J be ideals of the Γ- ring M and $f: M \to M/J$, the canonical homomorphism. Then $I + J = f^{-1}(f(I))$ and $(I + J)/J$ is Γ- isomorphic to $I/(I + J)$.

Now we define M - Module.

Let M be a Γ - ring. The additive abelian group N is said to be an **M – Module** if there is a mapping (composition $N \times \Gamma \times M \to N$ (or $M \times \Gamma \times N \to N$) (sending (n, γ, m) to $n\gamma m$ (or, (m, γ, n) to $n \gamma m$) such that

(i) $n\gamma(m_1 + m_2) = n\gamma m_1 + n\gamma m_2$

$n(\gamma_1 + \gamma_2)m = n\gamma_1 m + n\gamma_2 m$

$(n_1 + n_2)\gamma m = n_1\gamma m + n_2\gamma m$

(ii) $(n\gamma_1 m)\gamma_2 m = n\gamma_1(m\gamma_2 m)$, for every $n_i \in N$, $m_i \in M$ and $\gamma_i \in \Gamma$.

A submodule of an M - Module N is an additive subgroup S of N such that $S\Gamma M \subseteq S$.

N is said to be an irreducible M - Module if $N \Gamma M \neq (0)$ and if the only submodules of N are (0) and N.

2.27 Radical : A non - empty class $\mathfrak{R}$ of Γ- rings is a radical if and only if $\mathfrak{R}$ satisfies the following conditions :

(i) $\mathfrak{R}$ is homomorphically closed , i. e. , if A is in $\mathfrak{R}$ and I is an ideal of A , then A / I is in $\mathfrak{R}$.

(ii) $\mathfrak{R}$ is closed under extensions , i. e., for a Γ- ring A and an ideal I of A, both I and A / I are in $\mathfrak{R}$, then A is in $\mathfrak{R}$.

(iii) If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots.$ is an ascending chain of $\mathfrak{R}$ - ideals of a Γ- ring A , then $\cup_\alpha I_\alpha$ is in $\mathfrak{R}$.

2.28 Hereditary radical : Let α be a radical . Let a Γ- ring $M \in \alpha$ and I be an ideal of M . If $I \in \alpha$, then α is said to be hereditary radical .

2.29 Jacobson radical : The class of all right quasi - regular Γ- ring is a radical . The radical is called **Jacobson radical** and is denoted by J .

2.30 Residue class Γ-ring : Let M be a Γ- ring and I be an ideal of M . Define $M / I = \{ m + I \mid m \in M \}$, the set of cosets of I forms a Γ- ring with respect to the operations

$$(m + I) + (n + I) = (m + n) + I$$

and $(m + I)\gamma(n + I) = m\gamma n + I$.

We call M / I, the **residue class Γ- ring** of M with respect to I .

2.31 Strongly nilpotent Γ-ring : A subset S of M is strongly nil if each of its elements is strongly nilpotent . S is **strongly nilpotent** if there exists a positive integer n such that $(S\Gamma)^n S = (S\Gamma S\Gamma \ldots\ldots\ldots S\Gamma)S = 0$. Clearly a strongly nilpotent set is also strongly nil.

An one sided ideal I of a Γ- ring M is strongly nilpotent if $I^n = (I\Gamma I \Gamma\ldots\ldots I\Gamma)I = 0$, for some integer n.

2.32 Theorem : Let M be a Γ- ring. If M has no non-zero strongly nilpotent left ideals, then M has no non-zero strongly nilpotent right ideals.

Proof : Let I be a non-zero strongly right ideals of M. Then $I^n = (I\Gamma I \Gamma \ldots\ldots\ldots I\Gamma)I = 0$. Then $J = I + M\Gamma I$ is a left ideal of M. By induction of k, it can be shown that $J^k \subseteq I^k + M\Gamma I^k$, and hence $J^n \subseteq I^n + M\Gamma I^n = 0$, So J is a non -zero strongly nilpotent left ideal of M. Hence completed the proof.

2.33 Theorem : If M is a Γ- ring in the sense of Nobusawa and $a \in M$, then the following are equivalent :

(i) a is strongly nilpotent

(ii) $\langle a \rangle$ is strongly nil

(iii) $\langle a \rangle$ is strongly nilpotent.

We denote the strongly nilpotent radical by S(M).

2.34 Theorem : If A and B are strongly nilpotent ideals of a Γ- ring M, then their sum is a strongly nilpotent ideal of M.

Proof : Let A and B are strongly nilpotent ideals of a Γ- ring M, then if $(A\Gamma)^n A = 0$. Then $((A+B)\Gamma)^n (A+B) = (A\Gamma)^n A + B_1 = B_1$, where $B_1 \subseteq B$. If $(B\Gamma)^m B = 0$ then, $((A+B)\Gamma)^{mn+m+n}(A+B)$

$= (((A+B)\,\Gamma)^{n}(A+B)\,\Gamma)^{m}((A+B)\,\Gamma)^{n}(A+B) = (B_1\,\Gamma)^{m}B_1$

$= 0$.

Hence A+B is strongly nilpotent .

2.35 Theorem : If A and B are strongly nil ideals of a Γ- ring M , then their sum is a strongly nil ideal of M .

Proof : The proof of this theorem is similar to that of the previous theorem .

2.36 Theorem : If M is a Γ- ring , then S(M) is a strongly nil ideal of M .

Proof : Let $x \in S(M)$ then x is in a finite sum of strongly nilpotent ideals of M ,which by the previous theorem is strongly nilpotent , whence S(M) is strongly nil . Hence completed the proof .

2.37 Theorem : An ideal Q in a Γ- ring M is a semiprime ideal in M if and only if the residue class Γ- ring M / Q contains no nonzero strongly nilpotent ideals .

2.38 Locally nilpotent : A subset S of a Γ- ring M is said to be **locally nilpotent** if for any finite set $F \subseteq S$ and any finite set $\Phi \subseteq \Gamma$, there exists a positive integer n such that $(F\Phi)^n F = 0$.

An ideal I of a Γ-ring M is said to be locally nilpotent , if it is locally nilpotent as a Γ- ring . By taking F = {x} and {γ} we see that a locally nilpotent set is nil .

The Leivitzki nil radical of M is the sum of all locally nilpotent ideals of M is denoted by L(M) .

2.39 Lemma : Every sub ring and every homomorphic image of a locally nilpotent Γ- ring is locally nilpotent .

2.40 Theorem : Let M be a Γ- ring and I be an ideal of M such that both I and M / I are locally nilpotent Γ- ring . Then M is locally nilpotent Γ-ring .

Proof : Let S be a finite subset of M and suppose S $=\{s_1,s_2,s_3,\ldots\ldots\ldots\ldots,s_r\}$. Also consider that $s_i + I$, $i = 1,2 , \ldots\ldots, r$ are finite number of cosets of M / I . The subring generated by the cosets is S which is finite and and also subset of M / I . Since M /A is locally nilpotent , then by the definition for any finite subset $\Phi \subseteq \Gamma$, there exists a positive integer n such that $(S\Phi)^n S = 0 \in M / I$. Therefore, $(S\Phi)^n S \subseteq I$.

Now $(S \Phi)^n S$ is generated by a finite set of elements namely the set of all products of n of $S_i \Phi_i$, $\forall \Phi_i \in \Phi$, with S_i .Since I is locally nilpotent Γ- ring , then there exists a positive integer m such that $((S\Phi)^n S \Phi)^m(S\Phi)^n S = 0$.

i, e . $(S \Phi)^{mn+m+n} S= 0$.

Hence M is a locally nilpotent Γ- ring .

2.41 Theorem : Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots\ldots\ldots\ldots$ is the ascending chain of locally nilpotent ideals of a Γ- ring M , then $U_\alpha I_\alpha$ is locally nilpotent Γ- ring .

Proof : Suppose S is a finite subset of M and let $S \subseteq U_\alpha I_\alpha$. Then S is contained in some I_α . Since I is locally nilpotent , then for any finite subset $\Phi \subseteq \Gamma$, there exists a positive integer n such that $(S \Phi)^n S = 0$. Therefore , $U_\alpha I_\alpha$ is a locally nilpotent Γ- ring . Therefore, the lemma is proved .

2.42 Theorem : If I is an ideal of a Γ-ring M, then $J(I) = I \cap J(M)$.

Proof : First we show that $I \cap J(M) \subseteq J(I)$.

Let $a \in I \cap J(M)$ and $\gamma \in \Gamma$. Then $a \in J(M)$. So that there exist $x_i \in M$, $\zeta_i \in \Gamma$ such that

$x\gamma a + \Sigma x \zeta_i x_i - \Sigma x \gamma a \zeta_i x_i = 0$, for all $x \in M$.

Then $x\gamma a \gamma a + \Sigma x \zeta_i x_i \gamma a - \Sigma x \gamma a \zeta_i x_i \gamma a = 0$

Since $a \in I$ and for each $x_i \gamma a \in I$, we see that a is rqr in I. So that $I \cap J(M)$ is a rqr ideal of I.

Now we have to show that $J(I) \subseteq I \cap J(M)$, for let $a \in J(I)$ and $b \in |a>$. Then for any $\gamma \in \Gamma$, $(b\gamma)^2 b$ is in the principal right ideal in I generated by a.

Hence $(b\gamma)^2 b$ is rqr in I. Therefore we have,

$y\gamma(b\gamma)^2 b + \Sigma y \delta_i y_j - \Sigma y \gamma (b\gamma)^2 b \delta_i y_j = 0$, for all $y \in I$, where $\delta_j \in \Gamma$, $y_i \in I$.
If $x \in M$, then $x\gamma b \in I$, so $(x\gamma b)\gamma(b)^2 b + \Sigma x \gamma b \delta_j y_j - \Sigma x \gamma b \gamma (b\gamma)^2 b \delta y_j$, or, $x(\gamma b)^4 + \Sigma x\gamma b \delta_j y_j - \Sigma x (\gamma b)^4 \delta_j y_j = 0$. This may be written as $x\gamma b + (\Sigma x(\gamma b)^3 \delta_j y_j + \Sigma x (\gamma b)^2 \delta_j y_j + \Sigma x(\gamma b)\delta_j y_j - x(\gamma b)^3 - x(\gamma b)^2 - x\gamma b) - (x(\gamma b)^4 \delta_j y_j + \Sigma x(\gamma b)^3 \delta_j y_j + \Sigma x(\gamma b)^2 \delta_j y_j\, x(\gamma b)^4 - x(\gamma b)^3 - x(\gamma b)^2) = 0$, which is of the form

$x\gamma b + \Sigma x \lambda_k z_k - \Sigma x \gamma b \lambda_k z_k = 0$, Hence b is rqr in M, whence $|a>$ is rqr in M, Therefore $a \in J(M)$.

2.43 Direct summand : Let M be a Γ-ring. An ideal A of M is called a **direct summand** if there exists an ideal B of M such that every element x of M is uniquely expressible by $x = a + b$, $a \in A$, $b \in B$. We will write $M = A \oplus B$. Also if $a \in A$, $b \in B$, then $a\gamma b = 0$, for all $\gamma \in \Gamma$.

2.44 m-system : A subset S of M is an **m-system** in M if $S = \Phi$ or if $a, b \in S$ implies $\langle a\rangle \Gamma \langle a\rangle \cap S \neq \Phi$. The prime radical of M, which we denote by P (M), is defined as the set of elements x in M such that every m-system containing x contains 0 . Barnes has characterized P (M) as the intersection of all prime ideals of M has shown that an ideal P is a prime if and only if its complement P' is an m-system and that an ideal P of a Γ-ring M in the sense of Nobusawa is prime if and only if $a \Gamma b \subseteq P$ implies that $a \in P$ or $b \in P$.

A subset N of M is said to be an n - system in M if $N = \Phi$ or if $a \in N$ implies $\langle a\rangle \cap N \neq \Phi$.

2.45 Lemma : If N is an n-system in a Γ - ring M and $a \in N$, then there exists an m - system L such that $a \in L$ and $L \subseteq N$.

2.46 Theorem : If I is an ideal of the Γ- ring M then $P(I) = I \cap P(M)$, where P(I) denotes the prime radical of I considered as a Γ- ring .

Proof : Let P(I) is the set of elements x in I such that every m - system of I which contains x contains 0 . Every m - system of I is certainly also an m - system of M . It follows that $P(I) \supseteq I \cap P(M)$. By the last lemma $P(I) \subseteq I \cap P(M)$. Thus $P(I) = I \cap P(M)$. Hence completed the proof .

CHAPTER - THREE

k – regular gamma rings

In this chapter we have defined a regular Γ- ring in the sense of Kyuno [16]. We have shown that the class of all regular Γ- rings in the sense of Kyuno is a radical. We have also developed some of the characterizations of these Γ- rings. For the simplicity of languages, we call k-regular instead of 'regular in the sense of Kyuno'.

3. Definition.

3.1 k - regular Γ- ring : A Γ- ring M is called a **k - regular** if for every $a \in M$, there exists $\gamma \in \Gamma$ such that $a \gamma a = a$

3.2 lemma : In a k - regular Γ- ring M, $a \gamma b + b \gamma a = 0$, $a, b \in M$, $\gamma \in \Gamma$.

Proof : We have $(a+b) = (a+b)\gamma(a+b)$

$$= a\gamma a + b\gamma a + a\gamma b + b\gamma b$$

$$= a + b\gamma a + a\gamma b + b$$

$$\Rightarrow \quad a\gamma b + b\gamma a = 0.$$

3.3 Lemma : For any $a \in M$, $a + a = 0$.

Proof : Since M is k – regular, we have $a\gamma a = a$ for any $a \in M$. Hence $(a+a)\gamma(a+a) = a + a$

That is, $a + a = a\gamma a + a\gamma a + a\gamma a + a\gamma a$

$= (a+a)+(a+a)$

$\Rightarrow \quad a + a = 0$

From the above lemma we can easily say that M is commutative. For

$$a\gamma b = -(b\gamma a) = (-b)\gamma a = b\gamma a, \text{ for } a, b \in M, \gamma \in \Gamma.$$

3.4 Theorem : For a Γ- ring M with a unity 1, the following statements are given :

(i) M is a k - regular Γ-ring.

(ii) Every principal left (right) ideal $M \Gamma a$ ($a \Gamma M$) of M is generated by an idempotent of the right (left) operator ring R(L).

(iii) For every principal left (right) ideal $M \Gamma a$ ($a \Gamma M$) of M there exists $b \in M$ such that $M = M\Gamma a \oplus M\Gamma b$.

(iv) Every principal left (right) ideal $M \Gamma a$ ($a \Gamma M$) of M is a direct summand of M. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof : (i) $\Rightarrow$ (ii) : Given $a \in M$, there exists $\gamma \in \Gamma$ such that $a\gamma a = a$.

Therefore $\gamma a \gamma a = \gamma a$

This implies that $[\gamma, a][\gamma, a] = [\gamma, a]$, and so that $[\gamma, a]$ is an idempotent of R. It is clear that $M\Gamma a \supseteq M\gamma a$

Since $M\Gamma a = M\Gamma a\gamma a \subseteq M\gamma a$

Therefore, $M\Gamma a = M\gamma a$

Thus the left ideal $< a| = M\gamma a$

(ii) $\Rightarrow$ (iii) :

Let $M\Gamma a = M\gamma e$, where $e\gamma e = e$

Since $1 = e + (1 - e)$, then $M\gamma 1 = M\gamma e + M\gamma(1-e)$ implies

$M = M\gamma e + M\gamma(1-e)$.

If $a, b \in M$ are such that $a\gamma e = b\gamma(1-e)$

then $a\gamma e = a\gamma e\gamma e = b\gamma(1-e)\gamma e$

$$= b\gamma(1\gamma e - e\gamma e) = b\gamma(e - e) = b\gamma 0 = 0$$

Hence $M = M\gamma a \oplus M\gamma(1\text{-}e)$

(iii) $\Rightarrow$ (iv) is trivial.

3.5 Theorem : If M is a k - regular Γ- ring , then every finitely generated left (or right) ideal is principal.

Proof : If $a, b \in M$ then we have to show that $M\Gamma a + M\Gamma b$ is principal.

Since M is a k – regular Γ- ring, every principal left ideal of M is generated by an idempotent of the right operator ring R. So we have to prove that $M\gamma e_1 + M\gamma e_2$ (with $[\gamma, e_1]$, $[\gamma, e_2]$ idempotent) of R) is principal.

Now, $M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma(e_2 - e_2\gamma e_1)$ for

$a\gamma e_1 + b\gamma e_2 = (a + b\gamma e_2)\gamma e_1 + b\gamma(e_2 - e_2\gamma e_1)$.

If $x \in M$ such that

$(e_2 - e_2\gamma e_1)\gamma x\gamma(e_2 - e_2\gamma e_1) = e_2 - e_2\gamma e_1$

then $e_2' = x\gamma(e_2 - e_2\gamma e_1)$ implies

$$e_2'\gamma e_2' = x\gamma(e_2 - e_2\gamma e_1)\gamma x\gamma(e_2 - e_2\gamma e_1) = x\gamma(e_2 - e_2\gamma e_1) = e_2',$$

so that e_2' is an idempotent and then

$M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma e_2'$ with $e_2'\gamma e_2 = x\gamma(e_2 - e_2\gamma e_1)\gamma e_1$

$$= x\gamma(e_2\gamma e_1 - e_2\gamma e_1\gamma e_1) = x\gamma(e_2\gamma e_1 - e_2\gamma e_1)$$

$$= 0.$$

Finally, $M\gamma e_1 + M\gamma e_2' = M\gamma(e_1 + e_2' - e_1\gamma e_2')$

because, $a\gamma e_1 + b\gamma e_2' = (a\gamma e_1 + b\gamma e_2')\gamma(e_1 + e_2' - e_1\gamma e_2')$.

Hence $M\Gamma a + M\Gamma b$ is a principal left ideal. Similarly $a\Gamma M + b\Gamma M$ is a principal right ideal.

3.6 Theorem : If M is a k - regular Γ- ring and if J is a two sided ideal, then M / J is a k - regular Γ- ring.

Proof : Let $\bar{a} \in M/J$. Then $\bar{a} = a + J, a \in M$

Then there exists $\gamma \in \Gamma$ such that $a = a\gamma a$

Hence $\bar{a}\gamma\bar{a} = (a+J)\gamma(a+J)$

$= a\gamma a + J = a + J = \bar{a}$

3.7 Theorem : Let M be a k - regular Γ- ring without zero divisors. Then for any non zero $a, b \in M$, $a\gamma b = 1$ for some $\gamma \in \Gamma$.

Proof : Let $a, b \in M, a \neq 0, b \neq 0$. Then there exist $\gamma, \delta \in \Gamma$, such that $a = a\gamma a$ and $b = b\delta b$.

Now $a\gamma(a\gamma b - b) = a\gamma a\gamma b - a\gamma b$

$= a\gamma b - a\gamma b$

$= 0$.

Hence $a\gamma b - b = 0$.

So $b\delta(a\gamma b - b) = 0$ implies $b\delta a\gamma b - b\delta b = 0$. Hence $b\delta a\gamma b - b = 0$ and so $b\delta(a\gamma b - 1) = 0$. Since $b \neq 0$, hence $a\gamma b - 1 = 0$, that is $a\gamma b = 1$.

3.8 Corollary : Every k - regular Γ- ring without zero divisors is a skew Γ - field.

3.9 Semi-hereditary : A k - regular Γ- ring M is right **semi - hereditary** if every finitely generated right ideal of M is a projective R-module.

A right ideal I in M is called essential if for every non - zero right ideal A in M, $I \cap A \neq 0$.

Let $\Phi(M)$ be the set of all essential right ideals in M, and $Z_r(M) = \{x \in M \mid x\Gamma I = 0$, for some $I \in \Phi(M)\}$. M is called a right non - singular Γ-ring if $Z_r(M) = 0$. Similarly a left semi-hereditary Γ-ring and a left non - singular Γ- ring are defined.

3.10 Theorem : Let M be a k - regular Γ- ring. Then

(i) all one sided ideals in M are idempotent,

(ii) all two sided ideals are semiprime,

(iii) the Jacobson radical of M is zero,

(iv) M with the left and right unities is right and left semi - hereditary,

(v) M is right and left non - singular.

Proof : (i) : Let J be a right ideal of M. For each $a \in J$ there exists $\gamma \in \Gamma$ such that $a = a\gamma a$.

Consequently, $a = a\gamma a \in J\Gamma J$,

that is $J \subseteq J\Gamma J$.

And so $J = J\Gamma J$. Thus we have (i).

(ii) : Let I be two sided ideal of M. If J is a two sided ideal in M such that $J\Gamma J \subseteq I$, then $J \subseteq I$, because by (i) $J = J\Gamma J$. Hence we have (ii).

(iii) : Suppose that e is a right quasi - regular and $e = e\delta e$. Then there exists $r \in R$ such that $[\delta, e] \circ r = r + [\delta, e] - [\delta, e] \circ r = 0$. It follows that

$[\delta, e] = [\delta, e] \circ 0 = [\delta, e] \circ ([\delta, e] \circ r) = ([\delta, e] \circ [\delta, e]) \circ r = [\delta, e] \circ r = 0$.
Thus $e = e \delta e = e [\delta, e] = e 0 = 0$.
Recall that $J(M) = \{ e \in M \mid <e> \text{ is right quasi regular} \}$. Since $<e> = 0$, $e = 0$ and so $J(M) = 0$.

(iv) : We know every finitely generated right ideal in M may be written as $h \gamma M$, where $h \gamma h = h$. Let $A = \{ x \in M \mid h \gamma x = 0 \}$. Clearly A is a right ideal in M. For any $x \in M$, $x = h \gamma x + (x - h \gamma x)$, and $M = h \gamma M \oplus A$, because if $a \in h \gamma M \cap A$, then $a = h \gamma a = 0$. Thus $h \gamma M$ is a direct summand of M and so every finitely generated right ideal in M is a projective R – module . It can be proved that M is left semi – hereditary .

(v) Let J be an essential right ideal of M . Suppose that $a \gamma J = 0$, for some $a \in M$, and that there exists $\gamma \in \Gamma$ such that $a = a \gamma a$. Then
$a \gamma M \cap J = 0$ for if $x \in a \gamma M \cap J$, then $x = a \gamma x = 0$.
Since J is essential , $a \gamma M = 0$ and so $a = 0$.

3.11 Theorem : If M is a k - regular Γ- ring , then every two sided ideal J of M is the intersection of maximal left ideals (and also of maximal right ideals) .

Proof : Here M is a k - regular Γ- ring . Hence M / J is a k - regular Γ- ring by Theorem 3.6 .So $J R(M / J) = 0$, by Theorem 3.10 (iii). Therefore , J is equal to the intersection of the maximal left ideals of M . The proof for the maximal right ideals is similar .

3.12 Theorem : $M \gamma a = a \gamma M$, for all $a \in M$ .That is every left (or right) ideal is a two sided ideal .

Proof : Let $a \in M$. Then for every $b \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma b = b\gamma a$. So clearly we say that $M\gamma a = a\gamma M$.

3.13 Theorem : If a Γ-ring M is k - regular, then

(i) M is semi-prime .

(ii) The union of any chain of semi - prime ideal of M is semi - prime .

(iii) M/P are k - regular for all prime ideals P of M.

Proof : Let M be k - regular . then all ideals of M are semi – prime, whence (i) and (ii) hold . (iii) obviously holds for

$$(x+P)\gamma(x+P) = x\gamma x+P = x+P.$$

3.14 Weakly nilpotent : An element $a \in M$ is said to be a **weakly nilpotent** element if there exists a non-zero element $\gamma \in \Gamma$ and an integer n > 1 such that $(a\gamma^{n-1})a = 0$. A Γ-ring M is weakly nilpotent if every element of M is weakly nilpotent .

3.15 Theorem : In a Γ-ring M with no non - zero weakly nilpotent elements , every idempotent commutes with every element in M.

Proof : Let $e = e\gamma e, \gamma \in \Gamma$, and $x \in M$. If $e = 0$, then $x\gamma e = 0 = e\gamma x$. Suppose $e \neq 0$, then $\gamma \neq 0$. Since

$$(e\gamma x - e\gamma x\gamma e)\gamma(e\gamma x - e\gamma x\gamma e)$$

$$= (e\gamma x\gamma e - e\gamma x\gamma e\gamma e)([\gamma, x] - [\gamma, x\gamma e])$$

$$= (e\gamma x\gamma e - e\gamma x\gamma e)([\gamma, x] - [\gamma, x\gamma e])$$

$= 0$, and M has no non – zero weakly nilpotent elements , so $e\gamma x - e\gamma x\gamma e = 0$, or $e\gamma x = e\gamma x\gamma e$, Similarly $x \gamma e = e \gamma x \gamma e$, and so $e \gamma x = x \gamma e$,

3.16 Sub - directly irreducible : A gamma ring M is said to be **sub - directly irreducible** if the intersection of all non zero ideals of M is not zero .

3.17 Division gamma ring : A gamma ring M is said to be a **division gamma ring** if M has the strong left unity $[e, \delta]$ and the strong right unity $[\delta, e]$, and if for each non zero element $a \in M$ there exists $b \in M$ such that

$a \delta b = b \delta a = e$.

3.18 Theorem : A non zero sub - directly irreducible k - regular Γ- ring with no non - zero weakly nilpotent elements is a division Γ- ring .

Proof : Let M be a non - zero sub-directly irreducible k - regular Γ-ring with no non zero weakly nilpotent elements . For each non - zero element $a \in M$ there exists $\gamma \in \Gamma$ such that $a \gamma a = a$. For any $x \in M$ we have $a \gamma x = x \gamma a$. Let us consider two ideals $a \gamma M$ and $A = \{x - a \gamma x \mid x \in M\}$, whose intersection is zero . M is sub - directly irreducible , so $a \gamma M = 0$ or $A = 0$. But $a \gamma M \neq 0$. Hence $A = 0$, and thus $a \gamma x = x \gamma a$. This means that $[a, \gamma]$ and $[\gamma, a]$ are the strong left and right unities respectively . Let b be a non zero element of M . Then there exists $\delta \in \Gamma$ such that $b \delta b = b$. Then we can write $b \delta x = x = x \delta b$ for any $x \in M$, and so $b \delta a = a = a \delta b$, whence $(b \gamma a) \delta a = a = a \delta (a \gamma b)$, which implies that $b \gamma (a \delta a) = a = (a \delta a) \gamma b$.

Therefore, M is a division Γ- ring .

3.19 Theorem : Let I be a two sided ideal of a Γ- ring M . If I and M / I are k - regular , then M is k - regular .

Proof : Let $x \in M$. Since M / I is k - regular , then $x \gamma x - x \in I$, since I is k - regular the ideal $< x \gamma x - x >$ (generated by $x \gamma x - x$) is equal to the intersection of maximal ideal of I . Since intersection of maximal ideals of I is equal to zero , then $< x \gamma x - x >$ is contained in zero . This implies that $x \gamma x = x$. Hence x is k - regular . That is M is k - regular .

Thus we have the following :

3.20 Theorem : The class of all k - regular Γ- rings is a radical .

CHAPTER - FOUR

von Neumann regular gamma rings

The object of this chapter is to characterize the von Neumann regular gamma rings. Here we have shown that the class of all von Neumann regular gamma rings is a radical. Some of the characterizations of these Γ- rings are developed.

4. Definition.

4.1 von Neumann regular gamma ring : Let M be a Γ- ring. An element $a \in M$ is called **von Neumann regular** in M if $a \in a\gamma M\gamma a$, for some $\gamma \in \Gamma$. This means that there exists an $x \in M$ such that $a = a\gamma x\gamma a$.

A Γ- ring M is called a von Neumann regular Γ- ring if all its elements are von Neumann regular.

4.2 Lemma : If a is von Neumann regular in M, then $[a, \gamma]$ is von Neumann regular in L, where L is the left operator ring in M.

Proof : Since a is von Neumann regular in M, then $a \in a\gamma M\gamma a$. There exists an $x \in M$ such that $a = a\gamma x\gamma a$. This implies that

$$a\gamma = a\gamma x\gamma a\gamma,$$

and hence $[a, \gamma] = [a, \gamma][x, \gamma][a, \gamma]$

Therefore $[a, \gamma]$ is von Neumann regular in L.

4.3 Lemma : If $a, c \in M$, $a - c$ is von Neumann regular in M and c is in $a\gamma M\gamma a$, then a is von Neumann regular in M .

Proof : There are elements $x \in M$ and $\gamma \in \Gamma$ such that

$$a-c=(a-c)\gamma x\gamma(a-c)$$

$$=a\gamma x\gamma a-c\gamma x\gamma a-a\gamma x\gamma c+c\gamma x\gamma c.$$

This implies that $a=a\gamma x\gamma a-c\gamma x\gamma a-a\gamma x\gamma c+c\gamma x\gamma c+c$.

Since $c \in a\gamma M\gamma a$, put $c=a\gamma y\gamma a$.

Then we have ,

$$a=a\gamma x\gamma a-a\gamma y\gamma a\gamma x\gamma a-a\gamma x\gamma a\gamma y\gamma a+a\gamma y\gamma a\gamma x\gamma a\gamma y\gamma a+a\gamma y\gamma a.$$

$$=a\gamma(x\text{-}y\gamma a\gamma x-x\gamma a\gamma y+y\gamma a\gamma x\gamma a\gamma y+y)\gamma a$$

$=a\gamma x'\gamma a$, where $x'=x\text{-}y\gamma a\gamma x-x\gamma a\gamma y+y\gamma a\gamma x\gamma a\gamma y+y \in M$.

Therefore , a is von Neumann regular in M .

4.4 Definition : Let I be an ideal of a Γ - ring M . If every element of I is von Neumann regular , then I is von Neumann regular .

4.5 Lemma : If M is von Neumann regular and J is a two sided ideal of M , then M / J is von Neumann regular .

Proof : Let $\bar{a} \in M/J$. Then $\bar{a}=a+J$, $a \in M$.

Then there exist $x \in M$ and $\gamma \in \Gamma$ such that $a=a\gamma x\gamma a$.

Now $\bar{a}\gamma x\gamma\bar{a}=(a+J)\gamma x\gamma(a+J)$

$=a\gamma x\gamma a+J = a+J = \bar{a}$

Therefore , M / J is von Neumann regular .

4.6 Lemma : If M/I and I are von Neumann regular, then M is von Neumann regular.

Proof : We have, $a+I \in (a+I)\gamma M\gamma(a+I)$, for some $\gamma \in \Gamma$.
Then there exists an $x \in M$ such that

$$a+I = (a+I)\gamma x \gamma (a+I) = a\gamma x \gamma a + I.$$

This implies that $a - a\gamma x\gamma a \in I$.

Since I is von Neumann regular and since $a\gamma x\gamma a \in a\gamma M\gamma a$, then by Lemma 4.3, a is von Neumann regular.

4.7 Lemma : Let $I_1 \subseteq I_2 \subseteq$ ....be the ascending chain of von Neumann regular ideals, Then $\cup_\alpha I_\alpha$ is von Neumann regular.

Proof : It is obvious.

From the Lemmas 4.5, 4.6 and 4.7, we have the following :

4.8 Theorem : The class of all von Neumann regular Γ - rings is a radical.

The characterizations of von Neumann regular Γ - rings :

4.9 (a) Theorem : Let M be a Γ - ring with unity. The following statements are equivalent.

(i) M is a von Neumann regular Γ - ring.

(ii) Every principal left ideal $M\Gamma a$ is generated by an idempotent.

(iii) For every principal left ideal $M\Gamma a$ of M, there exists $b \in M$ such that $M = M\Gamma a \oplus M\Gamma b$.

(iv) Every principal left ideal $M \Gamma a$ is a direct summand of M.

Proof : (i) $\Rightarrow$ (ii) :

Given $a \in M$. Let $x \in M, \gamma \in \Gamma$ be such that $a = a\gamma x \gamma a$.

Then $M \Gamma a$ is generated by $x \gamma a$ which is an idempotent element, for

$(x \gamma a) \gamma (x \gamma a) = x\gamma (a\gamma x \gamma a) = x \gamma a$.

(ii) $\Rightarrow$ (iii) :

Let $M \Gamma a = M \gamma e$, where $e = e \gamma e$.

since $1 = e + (1 - e)$, then

$M \gamma 1 = M \gamma e + M\gamma (1 - e)$ implies

$M = M \gamma e + M \gamma (1- e)$.

If $a, b \in M$ are such that $a \gamma e = b\gamma (1- e)$,

then $a \gamma e = a \gamma e \gamma e = b\gamma (1- e) \gamma e = b\gamma (1 \gamma e - e \gamma e)$

$= b\gamma (e - e) = b\gamma 0 = 0$.

Hence $M = M \gamma e \oplus M\gamma (1 - e)$.

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i) : Given $a \in M$, there exists a left ideal J of M such that

$M = M \gamma a \oplus J$

Hence $1 = x \gamma a + b$, where $x \in M$, $b \in J$.

So that $a = a \gamma 1 = a\gamma x \gamma a + a \gamma b$

This implies that $a \gamma b = a - a\gamma x \gamma a \in M\gamma a \cap J = 0$

So $a - a\gamma x \gamma a = 0$

Hence $a = a \gamma x \gamma a$.

Therefore $a \in a \gamma M \gamma a$. Hence M is von Neumann regular.

4.9 (b) Theorem : Let M be a Γ - ring with unity . The following statements are equivalent :

(i) M is a von Neumann regular Γ - ring .

(ii) Every principal right ideal $a\Gamma M$ is generated by an idempotent .

(iii) For every principal right ideal $a\Gamma M$ of M , there exists $b \in M$ such that $M = a\Gamma M \oplus b\Gamma M$.

(iv) Every principal right ideal $a\Gamma M$ is a direct summand of M .

Proof : The proof is similar to the proof of Theorem 4.9 (a) .

4.10 Theorem : If M is von Neumann regular, then every finitely generated left (right) ideal is principal .

Proof : Let $a, b \in M$. Then we have to show that $M\gamma a + M\gamma b$ is principal . Since M is von Neumann regular, every principal left ideal is generated by some idempotent of M . So it is sufficient to prove that $M\gamma e_1 + M\gamma e_2$ is principal (with e_1 & e_2 idempotent) .

Now $M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma(e_2 - e_2\gamma e_1)$, for

$a\gamma e_1 + b\gamma e_2 = (a + b\gamma e_2)\gamma e_1 + b\gamma(e_2 - e_2\gamma e_1)$.

If $x \in M$ such that $(e_2 - e_2\gamma e_1)\gamma x\gamma(e_2 - e_2\gamma e_1) = e_2 - e_2\gamma e_1$,

then $e'_2 = x\gamma(e_2 - e_2\gamma e_1)$ is an idempotent and then

$M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma e'_2$

with $e'_2\gamma e_1 = x\gamma(e_2 - e_2\gamma e_1)\gamma e_1$

$= x\gamma(e_2\gamma e_1 - e_2\gamma e_1\gamma e_1)$

$= x\gamma(e_2\gamma e_1 - e_2\gamma e_1)$

$= x\gamma 0$

$= 0$.

Finally, $M\gamma e_1 + M\gamma e_2' = M\gamma(e_1 + e_2' - e_1\gamma e_2')$.

Because, $a\gamma e_1 + b\gamma e_2' = (a\gamma e_1 + b\gamma e_2')\gamma(e_1 + e_2' - e_1\gamma e_2')$.

Similarly we can prove that every finitely generated right ideal is principal.

4.11 Theorem : If M is von Neumann regular Γ- ring, then the intersection of any two principal left ideals (or right ideals) of M is principal.

Proof : It is enough to prove that if $a, b \in M$ then $M\gamma a \cap M\delta b$ is principal ideal.

To prove this we choose $e_1 = x\gamma a$ and $e_2 = y\delta b$, where $x, y \in M$ and $\gamma, \delta \in \Gamma$ are such that $a = a\gamma x\gamma a$, $b = b\delta\gamma\delta b$. Then e_1 and e_2 are idempotents and $M\gamma a = M\gamma e_1$, $M\delta b = M\delta e_2$.

Hence $M = M\gamma e_1 \oplus M\gamma(1-e_1) = M\delta e_2 \oplus M\delta(1-e_2)$

and $M\gamma e_1 = \mathrm{Ann}_M[(1-e_1)\gamma M] = \{x \in M \mid x\gamma(1-e_1)\gamma M = 0\}$,

$M\delta e_2 = \mathrm{Ann}_M[(1-e_1)\delta M] = \{x \in M \mid x\delta(1-e_2)\delta M = 0\}$.

Indeed obviously $M\gamma e_1 \subseteq \mathrm{Ann}_M[(1-e_1)\gamma M]$.

Conversely, if $x \in M$ and $x\gamma(1-e_1) = 0$, writing

$x = a_1\gamma e_1 + b_1\gamma(1-e_1)$ we have

$a_1\gamma e_1\gamma(1-e_1) + b_1\gamma(1-e_1)\gamma(1-e_1) = 0$

and $b_1\gamma(1-e_1) = 0$, hence $x = a_1\gamma e_1 \in M\gamma e_1$.

Thus $M\gamma e_1 \cap M\delta e_2 = \mathrm{Ann}_M[(1-e_1)\gamma M + (1-e_2)\delta M]$,

Now there exists $e_3 \in M$ such that

$(1-e_1)\gamma M + (1-e_2)\delta M = (1-e_3)\xi M$, $\xi \in \Gamma$, and from

$M \xi e_3 = \mathrm{Ann}_M [(1 - e_3) \xi M]$, we deduce that $M \gamma e_1 \cap M \delta e_2 = M \xi e_3$.
Similarly $a \gamma M \cap b \delta M$ is a principal right ideal .

4.12 Theorem : The Jacobson radical of a von Neumann regular Γ - ring M is equal to zero .

Proof : Let $a \in J(M)$. Thus $M \gamma a \subseteq J(M)$. Since $M \gamma a$ is generated by an idempotent element e , $M \gamma a = M \gamma e$, and thus from $e \in J(M)$ it follows that $(1 - e)$ is invertible .

So there exists $x \in M$ such that $1 = x \gamma (1 - e) = x \gamma 1 - x \gamma e$

$$= x - x \gamma e .$$

Hence $e = 1 \gamma e = (x - x \gamma e) \gamma e = x \gamma e - x \gamma e \gamma e = x \gamma e - x \gamma e = 0$.
Therefore , $a = 0$.

4.13 Theorem : If M is a von Neumann regular Γ - ring , then every two sided ideal I of M is the intersection of maximal left ideals (and also of maximal right ideals) .

Proof : Since M is a von Neumann regular Γ - ring , hence M / I is a von Neumann regular Γ - ring . So we have $J(M / I) = 0$. Therefore I is equal to the intersection of the maximal left ideals of M .

4.14 Theorem : The center of a von Neumann regular Γ - ring M is von Neumann regular .

Proof : Let $a \in C(M)$ (center of M). Let $x \in M$ and $\gamma \in \Gamma$ be such that $a = a \gamma x \gamma a$.

$$\therefore a = a\gamma x\gamma a = a\gamma a\gamma x = x\gamma a\gamma a$$

so, $a\gamma x = a\gamma a\gamma x\gamma x$,

or, $a = a\gamma x\gamma a = (a\gamma a\gamma x\gamma x)\gamma a$

$$= a\gamma\ (a\gamma x\gamma x)\gamma a.$$

Now $a\gamma x \in C(M)$, because if $y \in M$ then

$$(a\gamma x)\gamma y = (x\gamma y)\gamma a = (x\gamma y)\gamma(a\gamma x\gamma a)$$

$$= (a\gamma x)\gamma y\gamma(a\gamma x) = a\gamma x\gamma a\gamma y\gamma x$$

$$= a\gamma y\gamma x = y\gamma(a\gamma x).$$

Also $a\gamma x\gamma x \in C(M)$, because

$$(a\gamma x\gamma x)\gamma y = (a\gamma x)\gamma(x\gamma y) = (x\gamma y)\gamma(a\gamma x)$$

$$= x\gamma y\gamma a\gamma x = x\gamma a\gamma y\gamma x = (a\gamma x)\gamma(y\gamma x)$$

$$= y\gamma(a\gamma x)\gamma x = y\gamma(a\gamma x\gamma x).$$

Hence the center of M is a von Neumann regular Γ - ring .

4.15 Theorem : Every von Neumann regular Γ - ring M without zero – divisors is a skew Γ- field .

Proof : Let $a \in M$, $a \neq 0$. Let $x \in M$ and $\gamma \in \Gamma$ be such that $a = a\gamma x\gamma a$. Then $a\gamma(x\gamma a - 1) = 0$, $(a\gamma x - 1)\gamma a = 0$, and hence $x\gamma a = 1$, $a\gamma x = 1$, and so a is invertible . Hence M is a skew Γ-field .

4.16 Theorem : If M is a von Neumann regular Γ - ring whose only nilpotent element is zero , then

α) Every idempotent element of M is in the center .

β) If $a \in M$, $a \neq 0$, then there exist $b \in M$, $\gamma \in \Gamma$ such that $a\gamma b = b\gamma a = f$ is idempotent and $a\gamma f = f\gamma a = a$.

δ) $M\gamma a = a\gamma M$ for all $a \in M$; hence every left (or right) ideal is a two sided ideal.

Proof : α) : Let $e \in M$ be idempotent . Let $a \in M$ be an arbitrary element and assume that zero is the only nilpotent element of M.

Since,
$$[(1-e)\gamma a\gamma e]\gamma[(1-e)\gamma a\gamma e]$$
$$=(1-e)\gamma a\gamma e\gamma(1-e)\gamma a\gamma e$$
$$=(1\gamma a\gamma e-e\gamma a\gamma e)\gamma(1\gamma a\gamma e-e\gamma a\gamma e)$$
$$=(a\gamma e-e\gamma a\gamma e)\gamma(a\gamma e-e\gamma a\gamma e)$$
$$=a\gamma e\gamma a\gamma e-e\gamma a\gamma e\gamma a\gamma e-a\gamma e\gamma e\gamma a\gamma e+e\gamma a\gamma e\gamma e\gamma a\gamma e$$
$$=a\gamma e\gamma a\gamma e-e\gamma a\gamma e\gamma a\gamma e-a\gamma e\gamma a\gamma e+e\gamma a\gamma e\gamma a\gamma e$$
$$=0$$

Again, $[e\gamma a\gamma(1-e)]\gamma[e\gamma a\gamma(1-e)]$
$$=(e\gamma a\gamma 1-e\gamma a\gamma e)\gamma(e\gamma a\gamma 1-e\gamma a\gamma e)$$
$$=(e\gamma a-e\gamma a\gamma e)\gamma(e\gamma a-e\gamma a\gamma e)$$
$$=e\gamma a\gamma e\gamma a-e\gamma a\gamma e\gamma e\gamma a-e\gamma a\gamma e\gamma a\gamma e+e\gamma a\gamma e\gamma e\gamma a\gamma e$$
$$=e\gamma a\gamma e\gamma a-e\gamma a\gamma e\gamma a-e\gamma a\gamma e\gamma a\gamma e+e\gamma a\gamma e\gamma a\gamma e$$
$$=0.$$

We have, $0=(1-e)\gamma a\gamma e=1\gamma a\gamma e-e\gamma a\gamma e$
$$=a\gamma e-e\gamma a\gamma e,$$
and $0=e\gamma a\gamma(1-e)=e\gamma a\gamma 1-e\gamma a\gamma e$
$$=e\gamma a-e\gamma a\gamma e.$$

Hence, $a\gamma e = e\gamma a\gamma e = e\gamma a$ and so e is in the center of M.

β) : Let M be a von Neumann regular Γ - ring having 0 as the only nilpotent element . Given $a \in M$, $a \neq 0$. Let $x \in M$ be such that $a\gamma x\gamma a = a$, for some $\gamma \in \Gamma$.

Then $e = a\gamma x$, $e' = x\gamma a$ are idempotent elements of M ; so e and e' belong to the center, and $f = e\gamma e'$ is an idempotent.

It follows that $a\gamma(x\gamma x\gamma a) = (a\gamma x)\gamma(x\gamma a) = e\gamma e'$.

Also $(x\gamma x\gamma a)\gamma a = [x\gamma\ (x\gamma a)]\gamma a$

$$= [(x\gamma a)\gamma x]\gamma a = [x\gamma(a\gamma x)]\gamma a$$

$$= [(a\gamma x)\gamma x]\gamma a = (a\gamma x)\gamma(x\gamma a)$$

$$= e\gamma e' .$$

Moreover, $a\gamma e\gamma e' = e\gamma a\gamma e' = a\gamma x\gamma a\gamma e' = a\gamma e' = a\gamma x\gamma a = a$;

$$e\gamma e'\gamma a = e\gamma a\gamma e' = e\gamma a\gamma x\gamma a = e\gamma a = a\gamma x\gamma a = a ;$$

$\delta)$: Given $y \in M$, we have $y\gamma a = (y\gamma a)\gamma e\gamma e'$

$$= e\gamma(y\gamma a)\gamma e'$$

$$= a\gamma x\gamma y\gamma a\gamma e' ,$$

and so there exists $z \in M$ such that $y\gamma a = a\gamma z$. This shows that $M\gamma a \subseteq a\gamma M$, and the converse is proved in a similar way.

Hence, since every left ideal J is the sum of the principal left ideals generated by its elements, J is also a right ideal and vice versa.

4.17 Corollary : Let M be a von Neumann regular Γ - ring. Then,

(i) All one sided ideals in M are idempotent.

(ii) All two sided ideals in M are semi - prime.

(iii) The Jacobson radical of M is zero.

(iv) M is right and left semi hereditary

(v) M is right and left non- singular .

Proof : (i) Let J be a right ideal of M. Since M is von Neumamn regular, for each $a \in J$, $a = a\gamma x \gamma a$ for $x \in M$, $\gamma \in \Gamma$.

Consequently, $a = a\gamma x\gamma a \in J\Gamma M\Gamma J$ and also $J\Gamma M\Gamma J \subseteq J\Gamma J$. That is $J \subseteq J\Gamma J$. Also $J\Gamma J \subseteq J$. Hence $J = J\Gamma J$.

(ii) Let I be a two sided ideal of M. If A is a two sided ideal in M such that $A\Gamma A \subseteq I$, then we have to show that $A \subseteq I$. Now by (i) $A = A\Gamma A \subseteq I$.

(iii) Suppose that e is right quasi von Neumann regular. Then $e = e\delta x\delta e$ for some $x \in M$, $\delta \in \Gamma$. Let R be a right operator ring of M. Then there exists $r \in R$, such that $[\delta, e] \circ r = r + [\delta, e] - [\delta, e] r = 0$

It follows that
$$\begin{aligned}[\delta, e] &= [\delta, e] \circ 0 \\ &= [\delta, e] \circ ([\delta, e] \circ r) \\ &= ([\delta, e] \circ [\delta, e]) \circ r \\ &= [\delta, e] \circ r = 0.\end{aligned}$$

Thus $e = e\delta x\delta e = e\delta x[\delta, e] = e\delta x 0 = 0$.

Recall that $J(M) \doteq \{ e \in M \mid \langle e \rangle \text{ is right quasi von Neumann regular} \}$.

Since $e = 0$, $\langle e \rangle = 0$ and so $J(M) = 0$.

Note that in Theorem 4.12, this was proved by another method.

(iv) According to 4.10 every finitely generated one-sided ideal of M is a direct summand of M and so is projective.

(v) Suppose that $x\gamma J = 0$ for some $x \in M$ and some $J \subseteq M$. There is an idempotent $e \in M$ such that $M\gamma e = M\gamma x$, and since $M\gamma e\gamma J = M\gamma x\gamma J$

$=0$. We see that $J \subseteq (1-e)\gamma M$. Then $J \cap e\gamma M = 0$, whence $e\gamma M = 0$, and so $x = 0$. Thus M is nonsingular .

4.18 Theorem : Any finite subdirect sum of von Neumann regular Γ - rings is von Neumann regular .

Proof : It suffices to show that a subdirect sum of two von Neumann regular Γ - rings is von Neumann regular .

Suppose that M has two ideals J and K such that $J \cap K = 0$

Now $\frac{J+K}{J}$ is an ideal of $\frac{M}{J}$

Since $\frac{J+K}{J} \cong \frac{K}{J \cap K}$ and since $\frac{K}{J \cap K}$ is von Neumann regular, then $\frac{J+K}{J}$ is von Neumann regular.

Since $\frac{J+K}{J}$ and J are von Neumann regular , then $J+K$ is von Neumann regular.

4.19 Theorem : In a von Neumann regular Γ - ring M with no non-zero weakly nilpotent elements, every idempotent commutes with every elements in M .

Proof : Let $e\delta e = e$, $\delta \in \Gamma$. Let $x \in M$. If $e = 0$, then $e\delta x = x\delta e$.

Suppose $e \neq 0$. Then $\delta \neq 0$.

Now , $(e\delta x - e\delta x\delta e)\ \delta\ (e\delta x - e\delta x\delta e)$

$$= (e\delta x\delta e - e\delta x\delta e)([\delta , x] - [\delta , x\delta e])$$

$$= 0 .$$

Therefore $(e\delta x - e)\delta(e\delta x - e) = 0$; and hence

$$e\delta x\delta e\delta x - e\delta e\delta x - e\delta x\delta e + e\delta e = 0$$

or, $e\delta x - e\delta x - e\delta x\delta e + e\delta e = 0$;

This implies $e\delta\,(e - x\delta e) = 0$.

Since $e \neq 0$, therefore, $e - x\delta e = 0$ and hence $e = x\delta e$.

Again, $(e\delta x\delta e - e\delta x)\ \delta\ (e\delta x\delta e - e\delta x)$

$= (e\delta x\delta e - e\delta x\delta e)\ \delta([\delta, x\delta e] - [\delta, x]) = 0$.

Therefore, $(e - e\delta x)\delta(e - e\delta x) = 0$

or, $e\delta e - e\delta x\delta e - e\delta e\delta x + e\delta x\delta e\delta x = 0$

or, $e\delta e - e\delta x\delta e - e\delta x + e\delta x = 0$

or, $(e - e\delta x)\delta e = 0$.

Since $e \neq 0$, Therefore, $e - e\delta x = 0$ and hence $e = e\delta x$.

Therefore, $e\delta x = x\delta e$.

4.20 Theorem : A non - zero subdirectly irreducible von Neumann regular gamma ring with no non - zero weakly nilpotent elements is a division gamma ring.

Proof : Let M be a non zero subdirectly irreducible von Neumann regular Γ - ring with no non - zero weakly nilpotent elements .

Theorem 4.19 shows that for any $x \in M$, $x\delta e = e\delta x$, where $e = e\delta e$.

Let $a \in M$, $a \neq 0$. Let us consider two ideals $a\delta M$ and $A = \{\, x - a\delta M \mid x \in M \,\}$, whose intersection is zero . M is subdirectly irreducible, so $a\delta M = 0$ or $A = 0$. But $a\delta M \neq 0$, hence $A = 0$, and thus $a\delta x = x$. So that we can write $x\delta e = e\delta x = x$. This means that $[e, \delta]$ and $[\delta, e]$ are the strong left and right unities respectively . Now we have $a\gamma x = x = x\gamma a$ for any $a, x \in M$, and so $a\gamma e = e = e\gamma a$, whence $(a\delta e)\gamma e = e = e\gamma(e\delta a)$, so that $a\delta(e\gamma e) = e = (e\gamma e)\delta e$. Therefore, M is a division gamma ring .

4.21 Lemma : If $x, y \in M$, $\gamma \in \Gamma$ and $x' = x - x\gamma y\gamma x$, and if $x' = x'\gamma a\gamma x'$ for some $a \in M$, then $x = x\gamma b\gamma x$, for some $b \in M$.

Proof : $x = x' + x\gamma y\gamma x$

$$\begin{aligned}
&= x'\gamma a\gamma x' + x\gamma y\gamma x \\
&= (x - x\gamma y\gamma x)\gamma a\gamma(x - x\gamma y\gamma x) + x\gamma y\gamma x \\
&= x\gamma(a - a\gamma x\gamma y - y\gamma x\gamma a + y\gamma x\gamma a\gamma x\gamma y + y)\gamma x \\
&= x\gamma b\gamma x,
\end{aligned}$$

where, $b = a - a\gamma x\gamma y - y\gamma x\gamma a + y\gamma x\gamma a\gamma x\gamma y + y$.

4.22 Lemma : Let $J \subseteq K$ be two sided ideals in a Γ- ring M, then K is von Neumann regular if and only if J and K/J are both von Neumann regular .

Proof : If K is von Neumann regular then obviously K/J is von Neumann regular . Given $x \in J$, we have, $x\gamma y\gamma x = x$ for some $y \in K$.

Then $z = y\gamma x\gamma y$ is an element of J and

$x\gamma z\gamma x = x\gamma y\gamma x\gamma y\gamma x = x\gamma y\gamma x = x$.

Hence J is von Neumann regular .

Conversely, assume that J and K/J are both von Neumann regular .Given $x \in K$, it follows from the von Neumann regularity of K/J that $x - x\gamma y\gamma x \in J$ for some $y \in K$.

Consequently,

$$x - x\gamma y\gamma x = (x - x\gamma y\gamma x)\gamma z\gamma(x - x\gamma y\gamma x) \text{ for some } z \in J$$

so that,

$$\begin{aligned} x - x\gamma y\gamma x &= x\gamma z\gamma x - x\gamma z\gamma x\gamma y\gamma x - x\gamma y\gamma x\gamma z\gamma x + x\gamma y\gamma x\gamma z\gamma x\gamma y\gamma x \\ &= x\gamma(z - z\gamma x\gamma y - y\gamma x\gamma z + y\gamma x\gamma z\gamma x\gamma y)\gamma x \\ &= x\gamma w\gamma x, \text{ for some } w \in K. \end{aligned}$$

Therefore, K is von Neumann regular.

In particular, we can say that every two sided ideal in a von Neumann regular Γ- ring is von Neumann regular. On the other hand, if J is a two sided ideal in a Γ- ring M such that J and M / J are both von Neumann regular, then M is regular . This method of checking von Neumann regularity is quite useful when constructing examples.

4.23 Proposition : Any finite sub - direct product of von Neumann regular Γ- rings is regular.

Proof : It suffices to consider the case of a Γ- ring M which is a sub - direct product of two von Neumann regular Γ- rings. Then M has two sided ideals J and K such that $J \cap K = 0$ and M / J and M / K are both von Neumann regular. Since J is isomorphic to the two sided ideal $(J + K)/J$ in the regular Γ-ring M / K, then from 4.22, we have J is von Neumann regular. So that M / J is von Neumann regular and so M is von Neumann regular.

Note that a sub-direct product of infinitely many von Neumann regular Γ-rings, such as **Z** (set of integer), need not be von Neumann regular.

4.24 Proposition : Let M be a Γ- ring, and set $R = \{ x \in M \mid M\Gamma x\Gamma M$ is a von Neumann regular ideal $\}$. Then,

(a) R is a von Neumann regular two sided ideal of M.

(b) R contains all von Neumann regular two sided ideals of M.

(c) M/R has no non-zero von Neumann regular two sided ideal.

Proof : (a). Given $x, y \in R$, we see that $M\Gamma y\Gamma M$ and $(M\Gamma x\Gamma M + M\Gamma y\Gamma M)/ M\Gamma y\Gamma M$ are both von Neumann regular, whence from 4.22 $M\Gamma x\Gamma M + M\Gamma y\Gamma M$ is von Neumann regular. Thus, $M\Gamma x\Gamma M + M\Gamma y\Gamma M \subseteq R$, for all $x, y \in R$. Hence R is a two sided ideal. It is clear that R is von Neumann regular.

(b) is obvious, and then (c) follows from lemma 4.22.

In order to show that the Γ-ring of all $m \times n$ matrices over a von Neumann regular Γ-ring is von Neumann regular, we proceed via the following lemma, which is useful in other cases as well.

4.25 Lemma : Let $e_1, e_2, e_3, \ldots\ldots\ldots, e_n$ be orthogonal idempotents in a Γ-ring M such that $e_1 + e_2 + e_3 + \ldots\ldots\ldots + e_n = 1$. Then M is regular if and only if for each $x \in e_i\gamma M\gamma e_j$, there exists $y \in e_j\gamma M\gamma e_i$ such that $x\gamma y\gamma x = x$, $\gamma \in \Gamma$.

Proof : First assume that M is von Neumann regular and let $x \in e_i\gamma M\gamma e_j$. Then $x\gamma y\gamma x = x$, for some $y \in M$.

Now $x\gamma(e_j\gamma y\gamma e_i)\gamma x = x\gamma e_j\gamma y\gamma e_i\gamma x = x\gamma z\gamma x = x$, $z \in e_j\gamma M\gamma e_i$

Conversely, assume that for any $x \in e_i\Gamma M e_j$, there exists $y \in e_j\gamma M\gamma e_i$ such that $x\gamma y\gamma x = x$. We proceed by induction on n. Since the case $n = 1$ is trivial we begin with the case $n = 2$. First consider an element $x \in M$ such that $e_1\gamma x\gamma e_2 = 0$. There are elements $y \in e_1\gamma M\gamma e_1$ and $z \in e_2\gamma M\gamma e_2$ such that

$$(e_1\gamma x\gamma e_1)\gamma y\gamma(e_1\gamma x\gamma e_1) = e_1\gamma x\gamma e_2 \text{ and}$$

$$(e_2\gamma x\gamma e_2)\gamma z\gamma(e_2\gamma x\gamma e_2) = e_2\gamma x\gamma e_2, \text{ then}$$

$x\gamma(y+z)\gamma x = (e_1\gamma x\gamma e_1 + e_2\gamma x\gamma e_1 + e_2\gamma x\gamma e_2)\gamma(y+z)\gamma(e_1\gamma x\gamma e_1 + e_2\gamma x\gamma e_1 + e_2\gamma x\gamma e_2)$

$= e_1\gamma x\gamma e_1\gamma y\gamma e_1\gamma x\gamma e_1 + e_2\gamma x\gamma e_1\gamma y\gamma e_1\gamma x\gamma e_1 + e_2\gamma x\gamma e_2\gamma z\gamma e_2\gamma x\gamma e_1 + e_2\gamma x\gamma e_2\gamma z\gamma e_2\gamma x\gamma e_2$

$= e_1\gamma x\gamma e_1 + e_2\gamma x\gamma e_2 + e_2\gamma x\gamma(y+z)\gamma x\gamma e_1$

As a result, we see that the element $x' = x - x\gamma(y+z)\gamma x$ lies in $e_2\gamma M\gamma e_1$. Then $x'\gamma w\gamma x' = x'$, for some $w \in e_1\gamma M\gamma e_2$, whence $x\gamma v\gamma x = x$, for some $v \in M$.

Now consider a general element $x \in M$, and choose an element $y \in e_2\gamma M\gamma e_1$ such that $(e_1\gamma x\gamma e_2)\gamma y\gamma(e_1\gamma x\gamma e_2) = e_1\gamma x\gamma e_2$. Since $y \in e_2\gamma M\gamma e_1$, we see that $e_1\gamma x\gamma y\gamma x\gamma e_2 = e_1\gamma x\gamma e_2$, whence $e_1\gamma(x - x\gamma y\gamma x)\gamma e_2 = 0$. By the case above ,there exists an element $z \in M$ such that $(x - x\gamma y\gamma x)\gamma z\gamma(x - x\gamma y\gamma x) = x - x\gamma y\gamma x$, hence $x\gamma w\gamma x = x$, for some $w \in M$. Therefore , M is von Neumann regular.

Finally, let $n > 2$, and assume that the lemma holds for $n-1$ orthogonal idempotents . Setting $f = e_2 + \ldots\ldots\ldots\ldots + e_n$ and $g = e_1 + e_2 + \ldots\ldots\ldots\ldots + e_n$, we thus know that $f\gamma M\gamma f$ and $g\gamma M\gamma g$ are von Neumann regular. Consider any element $x \in e_1\gamma M\gamma f$. There exists $y \in e_2\gamma M\gamma e_1$ such that $(x\gamma e_2)\gamma y\gamma(x\gamma e_2) = x\gamma e_2$, so that $(x - x\gamma y\gamma x)\gamma e_2 = 0$. Then $x - x\gamma y\gamma x \in g\gamma M\gamma g$, whence $(x - x\gamma y\gamma x)\gamma z\gamma(x - x\gamma y\gamma x) = x - x\gamma y\gamma x$ for some $z \in g\gamma M\gamma g$. As a result, $x\gamma w\gamma x = x$ for some $w \in M$, hence we obtain $f\gamma w\gamma e_1 \in f\gamma M\gamma e_1$ such that $x\gamma(f\gamma w\gamma e_1)\gamma x = x$, Likewise , for any $x \in f\gamma M\gamma e_1$ there is some $t \in e_1\gamma M\gamma f$ such that $x\gamma t\gamma x = x$. Applying the case $n = 2$ to the orthogonal idempotents e_1 and f, we conclude that M is von Neumann regular. Therefore the induction works .

4.26 Lemma : A non - zero regular Γ- ring M is indecomposable (as a Γ- ring) if and only if its center is a Γ- field .

Proof : Assume that M is indecomposable . Let S denote the center of M , and let x be any non - zero element of S . Then by 4.14 $x\gamma y\gamma x = x$, for some $y \in S$.

Now, $x\gamma y\gamma x\gamma y = x\gamma y$. i..e., $x\gamma y$ is a non-zero central idempotent in M . Since M is indecomposable ,

$x\gamma y = 1$. Therefore , S is a Γ- field .

In particular , this lemma shows that the center of any prime von Neumann regular Γ- ring is a Γ- field .

4.27 Definition : P is a **projective left A - module** when the following property holds : if $f : M \to N$ is any epimorphism , and $g : P \to N$ a homomorphism , there exists a homomorphism $h : P \to M$ such that $g = f \circ h$.

4.28 Theorem : If A' is a finitely generated projective module over a von Neumann regular Γ-ring M , then $End_M(A)$ is a von Neumann regular Γ-ring .

Proof : According to 4 . 25 , $e\gamma M_n(M)\gamma e$ is von Neumann regular for any n and any idempotent $e \in M_n(M)$.

4.29 Definition : Let M be an A – module . Then M is a **free module** whenever it has a basis . Thus every element $x \in M$ may be written in one and only one way in the form $x = \sum_{s\in S} a_s s$ (where $a_s \in A$) .

Examples of free modules :

i) The zero module is free, with empty basis.

ii) Every Γ- ring M is a free left (right) M – module; the set consisting only of the unit element is a basis.

4.30Theorem : If A is a projective right module over a von Neumann regular Γ- ring M, then all finitely generated submodules of A are direct summand of A.

Proof : Let A be a submodule of a free right M – module F. Given any finitely generated submodule $B \subseteq A$, we infer that F has a finitely generated free direct summand G which contains B. It suffices to prove that B is a direct summand of G, for then B is a direct summand of F and hence also of A.

Choose a positive integer n such that B can be generated by n elements, and embed G in a finitely generated free right M-module H which has a basis with at least n elements. Then there exists $f \in End_M(H)$ such that $f \gamma H = B$. According to 4.28, $End_M(H)$ is von Neumann regular, hence there exists $g \in End_M(H)$ such that $f \gamma g \gamma f = f$, consequently, $f \gamma g$ is an idempotent endomorphism of H such that $f \gamma g \gamma H = f \gamma H = B$, whence B is a direct summand of H. Therefore, B is a direct summand of G.

4.31 Theorem : A Γ- ring M is von Neumann regular if and only if all right (left) M - modules are flat.

Proof : First assume that M is von Neumann regular. Let F be any free right M- module, and let K be any submodule of F. If A is any finitely generated submodule of K, then A is a direct summand of F by

4.30 , whence F / A is projective . Now F / K is the direct limit of the modules F / A , where A ranges over all finitely generated submodules of K . Thus , F / K is a direct limit of projective modules , whence F / K is flat .

Conversely , assume that all right M-modules are flat . Given $x \in M$, the flatness of $M / x\gamma M$ implies that the natural map $(M/(x\gamma M)) \otimes_M M\gamma x \to M/(x\gamma M)$ must be injective , i. e. , the map $M\gamma x/(x\gamma M\gamma x) \to M/(x\gamma M)$ is injective . Thus , $M\gamma x \cap x\gamma M = x\gamma M\gamma x$, and , consequently , $x \in x\gamma M\gamma x$. Thererfore , M is von Neumann regular .

4.32 Lemma : For a commutative Γ- ring M , the following conditions are equivalent :

(a) M is von Neumann regular .

(b) $M_{M'}$ is a Γ- field for all maximal ideals M' of M .

(c) M has no non-zero nilpotent elements and all prime ideals of M are maximal .

(d) All simple M – modules are injective .

Proof : (a) ⇒ (d) : Let M' be a maximal ideal of M , let J be an ideal of M , and let $f : J \to M/M'$ be a non - zero homomorphism.

Then $(M' \cap J)\Gamma(M' \cap J) = M' \cap J$

Now $M' \cap J = (M' \cap J)\Gamma(M' \cap J)$

$\subseteq J\Gamma M' \subseteq \ker f \subseteq J$

Hence $J \not\subset M'$.

Consequently , $x + y = 1$ for some $x \in M'$ and $y \in J$,

and we set $w = f(y) \in M/M'$.

Given any $a \in J$ we have $x\gamma a + y\gamma a = (x + y)\gamma a = 1\gamma a = a$.

$\Rightarrow a - y\gamma a = x\gamma a \in M'\Gamma J \subseteq \ker f$, whence $f(a - y\gamma a) = 0$

$\Rightarrow f(a) - f(y\gamma a) = 0$

$\Rightarrow f(a) = f(y\gamma a) = f(y)\gamma f(a) = w\gamma f(a)$

Therefore, f extends to a map $M \to M/M'$.

(d) $\Rightarrow$ (c) : We first claim that if M' is any maximal ideal of M, then $x \in x\Gamma M'$ for all $x \in M'$. If not, then $x\Gamma M / x\Gamma M' \neq 0$ for some $x \in M'$. Then $M/M' \cong x\Gamma M / x\Gamma M'$. Hence there exists an epimorphism

$f: x\Gamma M \to M/M'$.

Now f extends to a map $g: M \to M/M'$, and so $f(x\gamma M) \subseteq g(M') = 0$, which is false. Thus the claim holds.

Suppose that $x\gamma x = 0$ for some nonzero $x \in M$. The annihilator $J = \{m \in M \mid m\gamma x = 0\}$ is a proper ideal and so is contained in a maximal ideal M'. Since $x \in J \subseteq M'$, we have $x \in x\Gamma M'$ by the claim above. Then $x = x\gamma y$ for some $y \in M'$, and $(1-y)\gamma x = 1\gamma x - y\gamma x = x - x\gamma y = 0$

$\Rightarrow 1-y \in J \subseteq M'$, which is impossible. Thus M cannot have any nonzero nilpotent elements.

Now let P be a prime ideal of M, and let M' be a maximal ideal which contains P. Given any $x \in M'$, we have $x \in x\Gamma M'$ and so

$x\gamma(1-y) = x\gamma 1 - x\gamma y = x - x\gamma y = 0$ for some $y \in M'$.

Since $1-y \notin M'$, we also have $1-y \notin P$, whence $x \in P$. Thus $M' = P$. So that P is maximal.

(c) $\Rightarrow$ (b) : Since there are no prime ideals of M properly contained in M', we have see that $M'\Gamma M_{M'}$ is the only prime ideal of $M_{M'}$, whence $M'\Gamma M_{M'}$ is nil.

Given $x/s \in M'\Gamma M_{M'}$, we thus have $(x/s)^n = 0$ for some n, hence $t\gamma x^n = 0$ for some $t \in M - M'$. Then $(t\gamma x)^n = 0$ and so $t\gamma x = 0$, whence $x/s = 0$. Thus $M'\Gamma M_{M'} = 0$. So that $M_{M'}$ is a Γ- field.

(b) $\Rightarrow$ (a) : Let A be any M-module .For any maximal ideal M' of M, it follows from (b) that $A_{M'}$ is a flat $M_{M'}$-module, and consequently A is a flat M-module . According to 4.31 M is von Neumann regular.

4.33 Theorem : A Γ-ring M is von Neumann regular if and only if

(a) M is semiprime .

(b) The union of any chain of semiprime ideals of M is semiprime.

(c) M/P is von Neumann regular for all prime ideals P of M.

Proof : If M is von Neumann regular, then obviously (c) holds. In view of 4.17 (iv) all two sided ideals of M are semiprime, whence (a) and (b) hold.

Conversely, assume that (a), (b), (c) hold .If M is not von Neumann regular, then there is some $x \in M$ such that $x \notin x\gamma M\gamma x$. Note that 0 is a semiprime ideal of M such that $x \notin x\gamma M\gamma x + 0$. From (b) we see that there is a semiprime ideal J in M which is maximal with respect to the property $x \notin x\gamma M\gamma x + J$.

Now M/J is not von Neumann regular, hence by (c), J is not prime. Thus there exist two sided ideals A and B which properly contain J, such that $A\Gamma B \subseteq J$. Now set $K = \{m \in M \mid m\Gamma B \subseteq J\}$ and $L = \{m \in M \mid K\Gamma M \subseteq J\}$. As J is semiprime, K and L are semiprime. Since $(K \cap L)\Gamma(K \cap L) \subseteq K\Gamma L \subseteq J$, We have $K \cap L \subseteq J$. Clearly, $A \subseteq K$ and $B \subseteq L$, hence K and L properly contain J.

Because of the maximality of J, there exist elements $y, z \in M$ such that $x - x\gamma y\gamma x \in K$ and $x - x\gamma z\gamma x \in L$.

Now $x - x\gamma(y + z - y\gamma x\gamma z)\gamma x = (x - x\gamma y\gamma x) - (x - x\gamma y\gamma x)\gamma z\gamma x \in K$

$= (x - x\gamma z\gamma x) - x\gamma y\gamma(x - x\gamma z\gamma x) \in L$.

We see that $x \in x\gamma M\gamma x + (K \cap L) \subseteq x\gamma M\gamma x + J$ which is a contradiction. Therefore M must be von Neumann regular.

4.34 Corollary : A Γ- ring M is von Neumann regular if and only if all two sided ideals of M are idempotent and M / P is von Neumann regular for all prime ideals P of M .

4.35 Definition : A **completely prime ideal** in a Γ - ring M is a proper two sided ideal P such that M / P is an intregral domain (not necessarily commutative) .

4.36 Lemma : If M is a Γ- ring with no non zero nilpotent elements , then every minimal prime ideal of M is completely prime .

Proof : We first claim that if $a_1, a_2, a_3 \ldots\ldots . a_n \in M$ and $a_1 a_2 a_3 \ldots\ldots a_n = 0$, then the product of the a_i in any order is zero . To prove this , it suffices to show that if $x\gamma a\gamma b\gamma y = 0$ in M , then $x\gamma b\gamma a\gamma y = 0$. This is clear if $x = y = 1$, then $x\gamma a\gamma b\gamma y = 0$

$\Rightarrow 1\gamma a\gamma b\gamma 1 = 0$

$\Rightarrow a\gamma b = 0$.

and so $(b\gamma a)\gamma(b\gamma a) = b\gamma(a\gamma b)\gamma a = 0$.

whence $b\gamma a = 0 \Rightarrow x\gamma b\gamma a\gamma y = 0$

In case $x = 1$, then $(a\gamma b)\gamma y = 0 \Rightarrow y\gamma(a\gamma b) = 0$

$\Rightarrow y\gamma(a\gamma b\gamma a) = 0$

$\Rightarrow a\gamma b\gamma a\gamma y = 0$

$\Rightarrow b\gamma a\gamma b\gamma a\gamma y = 0$

$\Rightarrow (b\gamma a)\gamma(b\gamma a)\gamma y = 0$

$\Rightarrow (b\gamma a)\gamma y\gamma(b\gamma a) = 0$

$\Rightarrow (b\gamma a)\gamma y)\gamma(b\gamma a)\gamma y = 0$

$\Rightarrow b\gamma a\gamma y = 0$.

For the general case ,

$x\gamma(a\gamma b\gamma y) = 0$

$\Rightarrow \quad a\gamma b\gamma y\gamma x = 0$

$\Rightarrow \quad (b\gamma a\gamma y)\gamma x = 0$

$\Rightarrow \quad x\gamma(b\gamma a\gamma y) = 0$.

This establishes the claim .

Now let P be any minimal prime ideal of M . Recall that on m - system in M is a nonempty subset X such that $0 \notin X$ and whenever $x, y \in X$ there exists $n \in M$ such that $x\gamma n\gamma y \in X$. Then M - P is an m - system and we may choose a maximal m - system $X \supseteq M - P$. If Q is a two sided ideal of M , maximal among all two sided ideals disjoints from X , then Q is prime. Since Q is disjoint from M - P, we have $Q \subseteq P$ and thus Q = P , by minimality of P. As a result , P is disjoint from X , whence X = M - P . Thus M - P is a maximal m- system .

Set $Y = \{x_1 x_2 x_3 \ldots\ldots x_n \mid x_1, x_2, x_3, \ldots\ldots, x_n \in M - P\}$. If $0 \in Y$, then $x_1 x_2 x_3 \ldots\ldots x_n = 0$, for some $x_i \in M - P$. There exist $m_1, m_2, m_3 \ldots\ldots, m_{n-1} \in M$, such that $x_1\gamma m_1\gamma x_2\gamma m_2\gamma \ldots\ldots \gamma x_{n-1}\gamma m_{n-1}\gamma x_n \in X = M - P$. This implies $x_1\gamma m_1\gamma x_2\gamma m_2\gamma \ldots\ldots \gamma x_{n-1}\gamma m_{n-1}\gamma x_n \notin P$. Since $(x_1\gamma x_2\gamma x_3 \ldots \gamma x_n)\gamma(m_1\gamma m_2\gamma m_3\gamma \ldots\ldots \gamma m_{n-1}) = 0$, we see from the claim above that $x_1\gamma m_1\gamma x_2\gamma m_2\gamma \ldots\ldots \gamma x_{n-1}\gamma m_{n-1}\gamma x_n = 0$ which is impossible . Thus $0 \notin Y$, whence Y is an m - system . clearly $M - P \subseteq Y$. Hence by maximality of M - P , we obtain M - P = Y. Therefore M - P is multiplicatively closed . So that M / P is a domain .

4.37 Theorem : Let M be a Γ-ring with no non-zero nilpotent elements . Then M is von Neumann regular if and only if M / P is von Neumann regular for all completely prime ideals P of M .

Proof : Assume that M / P is von Neumann regular for all completely prime ideals . If P is any minimal prime ideal of M , then is completely prime by 4.36 . Hence M / P is an integral domain and so is a division Γ- ring . Consequently , we see that M / Q is a division Γ- ring for every prime ideal Q of M .

Since every semiprime ideal of M is an intersection of prime ideals , we infer that the set of semiprime ideals of M coincides with the set of those two sided ideals J such that M / J has no nonzero nilpotent element .

As a result , we see that the union of any chain of semiprime ideals of M must be semiprime . Therefore, M is von Neumann regular .

CHAPTER - FIVE

Jacobson Γ- rings and special Jacobson radicals

In this chapter we have defined Jacobson gamma ring and showed that it is hereditary. We have also studied Jacobson radical for gamma rings. It is shown that the Jacobson radical for gamma rings is a special class of radicals .

5 Definition .

5.1 Jacobson Γ- ring : A Γ- ring M is said to be a Jacobson Γ- ring if $J(M/A) = \wp(M/A)$, for every ideal A of M , where J and $\wp$ represent Jacobson radical and prime radical respectively .

5.2 Theorem : The class of Jacobson Γ- ring is hereditary .

Let M be a Jacobson Γ-ring and let I be an ideal of M . Then we have to prove that I is a Jacobson Γ-ring .

Let A be an ideal of M ,

if $A \subseteq I$, then we have

$$J(I/A) = (I/A) \cap J(M/A)$$

and $\wp(I/A) = (I/A) \cap \wp(M/A)$

Since M is a Jacobson Γ- ring .

Then , $J(M/A) = \wp(M/A)$

so , $J(I/A) = \wp(I/A)$. Hence I is a Jacobson Γ- ring .

If A is not contained in I , then $(A+I)/A$ is a nonzero ideal of M . Since $(A+I)/A \cong I/(I \cap A)$. hence $J((A+I)/A) = ((A+I)/A) \cap J(M/A)$ and $\wp((A+I)/A) = (A+I)/A \cap \wp(M/A)$.

Since $J(I/(I\cap A)) = I/(I\cap A)\cap J(M/A)$ and

$\wp(I/(I\cap A)) = I/(I\cap A))\cap\wp(M/A)$, therefore

$I/(I\cap A)\cap J(M/A) = I/(I\cap A)\cap\wp(M/A)$.

Hence $J(I/(I\cap A)) = \wp(I/(I\cap A))$

Therefore, I is a Jacobson Γ- ring.

5.3 Theorem : The extension M' is a Jacobson Γ- ring if and only if M is a Jacobson Γ- ring.

Proof : Let M' be a Jacobson Γ- ring. The Γ- ring M is isomorphic to an ideal of M' and so, by Theorem 5.2 M is a Jacobson Γ- ring.

Conversely, if M is a Jacobson Γ- ring, and M' is an extension of a Jacobson Γ-ring M by the Jacobson Γ- ring Z. Since radical classes are closed under such extensions, we have that M' is a Jacobson Γ- ring.

5.4 Corollary : In a Γ- ring M', $J(M') = J(M)'$.

Proof : By theorem 5.2 $J(M)'$ is a Jacobson Γ- ring and an ideal of M'. So $J(M)' \subseteq J(M')$. However $M'/J(M)' \cong M/J(M)$. So $J(M)' \supseteq J(M')$

$\therefore$ $J(M') = J(M)'$.

5.5 Matrix Γ- ring : Let M be a division Γ- ring and $M_n(M)$ denote the additive group of all $n\times n$ matrices whose entries are from M. Then $M_n(M)$ is a Γ- ring with $\Gamma = M_n(M)$, under the usual matrix multiplication. This is called the matrix Γ- ring.

5.6 Proposition : A Γ-ring M is a Jacobson Γ-ring if and only if , for any n , the matrix Γ-ring $M_n(M)$ is a Jacobson Γ-ring .

Proof : Suppose that M is a Jacobson Γ-ring . By theorem 5.3 M' is a Jacobson Γ-ring . Any homomorphic image of $M_n(M)$ is of the form $M_n(M')$, where M' is an image of (M') . The Γ-ring M' is Jacobson and so $\wp(M') = J(M')$.

Since $M_n(M')$ is the homomorphic image of $M_n(M)$,

then $\wp(M_n(M')) = J(M_n(M'))$. Thus $M_n(M')$ is a Jacobson Γ-ring .

Conversely , suppose that $M_n(M')$ is a Jacobson Γ-ring . By the preceding case $M_n(Z)$ is a Jacobson Γ-ring . So that $M_n(M')$ being an extension of $M_n(M)$ by $M_n(Z)$ is a Jacobson Γ-ring . If I is a prime ideal of M' , then $M_n(M'/I)$ is prime and so semi primitive . Thus M'/I is semiprimitive and M' is a Jacobson Γ-ring . Finally by theorem 5.2 M is a Jacobson Γ-ring .

5.7 Definition . $(M : R) = \{r \in R : R\Gamma r \subseteq M\}$

$(M : R)$ is a two sided ideal of R

A Γ-ring M is a right primitive Γ-ring if M contains a maximal right ideal M', such that $(M' : M) = 0 = \{m \in M : M\Gamma m \subseteq M'\}$.

5.8 Right primitive ideal : An ideal P of M is **right primitive ideal** if M / P is right primitive .

5.9 Special class of Γ-rings : A class M of Γ-rings is a special class of Γ-rings if it satisfies the following three conditions :

(i) Every Γ-ring in the class M is prime Γ-ring .

(ii) Every non zero ideal of a Γ-ring in M is itself a Γ-ring in M .

(iii) If A is a Γ- ring in M , and A is an ideal of a Γ- ring K , then K/A* is in M, where A* is the anhilator of A, i,e. $A^* = \{x \in K : x \Gamma A = A \Gamma x = 0\}$.

5.10 Prime Γ- ring : A gamma ring M is said to be completely prime if $a \Gamma b = 0$ implies $a = 0$ or $b = 0$.

We recall Barnes definition : Let M be a Γ- ring . An ideal P of M is prime if for all pair of ideals S and T of M , $S\Gamma T \subseteq P$ implies $S \subseteq P$ or $T \subseteq P$. A Γ- ring M is prime if the zero ideal is prime .

5.11 Theorem : Every primitive Γ- ring is prime .

Proof : Suppose that M is primitive Γ- ring and that I is a maximal right ideal such that $(I : M) = 0$. Then 0 is the only two sided ideal of M contained in I , for if A is an ideal of M , $A \subseteq I$, then $M\Gamma A \subseteq I$, $A \subseteq (I:M) = 0$. Thus if $B\Gamma C = 0$ for ideals B ,C of M and if $B \neq 0$, then $B \not\subset I$. Therefore $M = I + B$, for I is maximal right ideal .

Now $M\Gamma C = (I+B)\Gamma C$

$$= I\Gamma C + B\Gamma C \subseteq I .$$

Thus $C \subseteq (I : M) = 0$, therefore , $C = 0$, and M is prime Γ- ring .

5.12 Theorem : Every non zero ideal of a primitive Γ- ring is primitive .

Proof : Suppose that M is a primitive Γ- ring and that I is a maximal right ideal such that $(I : M) = 0$. We shall show that M contains a regular maximal right ideal I_1 such that $(I_1 : M) = 0$.

Let a not in I and let $I_1 = \{x \in M : a \Gamma x \in I\}$, Then I_1 is of course a right ideal of M . Also $I_1 \neq M$ for if $a \gamma M \subseteq I$, then we represent $M = I + (a)_r$, where $(a)_r$ is the right ideal generated by a . Since I is maximal and a is not in I , M can be so represented . Then $M \Gamma M = I \gamma M + (a)_r \gamma M \subseteq I$. Then $M \subseteq (I : M) = 0$, a contradiction . Thus $I_1 \neq M$.

Take any b not in I_1 , i .e. , $a \gamma b \notin I$ Then so above , $a \gamma b \gamma M \not\subset I$ for if $a \gamma b \gamma M \subseteq I$, we get $M \gamma M \subseteq I$ and $M \subseteq (I : M) = 0$. Since I is maximal and $a \gamma b \gamma M$ is a right ideal not contained in I , we have , $M = a \gamma b \gamma M + I$. Thus for any y in M , there exist elements c in M , i in I , such that

$a \gamma y = a \gamma b \gamma c + i$. Thus $a \gamma (y - b \gamma c)$ is in I . Thus $y - b \gamma c$ is in I_1 . Therefore every y is in $I_1 + b \gamma M$ or $M = I_1 + b \gamma M$. This proves that I_1 is a maximal right ideal of M .

5.13 Lemma : The Jacobson radical of any Γ- ring M is equal to (α) , the intersection of all the regular maximal right ideals of M ,

to (β), the intersection of all the regular maximal left ideals of M ,

to (γ), $\{x : x \gamma M$ is right quasi-regular for every m in $M\}$,

to (δ), $\{x : M \gamma x$ is left quasi regular for every m in $M\}$.

Proof : If x is in J. then xγm is in J for every m in M , and $x \gamma m$ is right quasi regular . Thus $J \subseteq \{x : x \gamma m$ is right quasi regular $\}$.

Now take any element x in (α) . This means that x is in every regular maximal right ideal of M . Either x is right quasi regular , or if it is not , then $\{ m + x \gamma m \} \neq M$.

Let M' be a maximal right ideal containing $\{ m + x \gamma m \}$, but not containing x . Then M' is regular , for $- x \gamma m - m = x \gamma (-m) + (-m)$ is in M' for every m in M . In this case $x \in M'$ and therefore $M' = M$. This is a contradiction and , consequently every x in (α) is right quasi regular . Since (α) is also a right ideal , it is a right quasi- regular right ideal . Thus $(\alpha) \subseteq J$.

Now take x to be any element of (γ) . Thus $x\,\gamma\,m$ is right quasi - regular for every m in M . Now either $x \in (\alpha)$ or , if not there exists a regular maximal right ideal M' . Since M' is maximal , the right ideal generated by M' and x is all of M . Thus $M = \{m' + x\,\gamma\,(m + i)\}$, where i is an integer . Let e be the left unit of M' , i .e., $e\,\gamma\,m - m$ is in M' , for every m of M . Then there exists an m' in M', m in M , i an integer , such that

$$-e = m' + x\,(m + i)$$

Then $-e\,\gamma\,e = m'\,\gamma\,e + x\,\gamma\,(m + i)\,\gamma\,e$.

Now $x\,\gamma\,(m + i)\,\gamma\,e$ is right quasi regular and thus there exists an element z such that $x\,\gamma\,(m + i)\,\gamma\,e + z + x\,\gamma\,(m + i)\,\gamma\,e\,\gamma\,z = 0$. Now $m'\,\gamma\,e\,\gamma\,z + x\,\gamma\,(m + i)\,\gamma\,e\,\gamma\,z = -e\,\gamma\,e\,\gamma\,z$.

Thus $m'\,\gamma\,e\,\gamma\,z - x\,\gamma\,(m + i)\,\gamma\,e - z + e\,\gamma\,e\,\gamma\,z = 0$.

Now $e\,\gamma\,t - t$ is in M' for every t . Thus $e\,\gamma\,e - e$ is in M' , $e\,\gamma\,e\,\gamma\,z - e\,\gamma\,z$ is in M' , $e\,\gamma\,z - z$ is in M' and therefore , $e\,\gamma\,e\,\gamma\,z - z$ is in M'. Also $m'\,\gamma\,e\,\gamma\,z$ is in M' , since M' is a right ideal . Therefore, $x\,\gamma\,(m + i)\,\gamma\,e$ is in M' . Therefore $-e\,\gamma\,e = m'\,\gamma\,e + x\,\gamma\,(m + i)\,\gamma\,e$ is in M'. Since $e\,\gamma\,e - e$ is in M' , e is in M' Then $e\,\gamma\,m$ and $e\,\gamma\,m - m$ are both in M' , $-m$ and m are in M' for every m , and $M' = M$. This is impossible . Therefore , x is in (α)

Thus $(\gamma) \subseteq (\alpha) \subseteq J \subseteq (\gamma)$ and $J = (\alpha) = (\gamma)$. Similarly $J = (\beta) = (\delta)$, and the lemma is established .

5.14 Theorem : The radical J = the intersection of all the right primitive (two sided) ideals of M .

Proof : Every regular maximal right ideal R of M contains a right primitive ideal , namely $(R : M)$. Therefore J, which is the intersection of the regular maximal right ideals, contains the intersection of the right primitive ideals of R.

Conversely, we shall show that J is contained in every right primitive ideal of M. Let P be a right primitive ideal of M. Then M/P contains a maximal right ideal R/P such that $(R/P, M/P) = 0$ or $(R : M) \subseteq P$

Now if R is regular, then, $J \subseteq R$. Also $J \subseteq (R : M)$, for $(R : M)$ is the largest ideal of M contained in R. Then $J \subseteq (R : M) \subseteq P$.

When R is not regular, since $P \subseteq R$, $(R : M) \subseteq R$, and we can at least conclude that $(R : M)$ is the largest ideal of M contained in R. For if Q is an ideal of M and $Q \subseteq R$ then $M \Gamma Q \subseteq Q \subseteq M$. Therefore $Q \subseteq (R : M)$. In particular, then $P = (R : M)$. Now if the radical J is not contained in P, then it cannot be contained in $(R : M)$ and so it cannot be contained in R, a contradiction. We can write $J \subseteq P$. Now if $R = \{ x : x \gamma M \subseteq R \}$. The right hand side of this equation is a right ideal which contains R. Since R is maximal, it is either R or M. If $\{ x : x \gamma M \subseteq R\} = M$, then $M \Gamma M \subseteq R$. Then $(R : M)$ and thus $R = M$, a contradiction.

Now assume that $J \not\subset M$ and take x in J, $x \notin R$. Then $M \gamma x$ is not in R, for otherwise x would be in $(R : M) \subseteq R$. Take an element z in M such that $z \gamma x \notin R$. Then $z \gamma x \gamma M \not\subset R$, since $\{ w : w \gamma M \subseteq R\} = R$. Since R is maximal, $R + z \gamma x \gamma M = M$, and thus there exists an $r \in R$ and m in M such that $r + z \gamma x \gamma m = -z$. Then $z + z \gamma x \gamma m$ is in R.

Since x is in J, $x \gamma m$ is right quasi – regular and therefore there exists an element w such that $x \gamma m + w + x \gamma m \gamma w = 0$.

Then, $z = z + z \gamma (x \gamma m + w + x \gamma m \gamma w)$

$= (-z + z \gamma x \gamma m) + (z + z \gamma x \gamma m) \gamma w$.

Thus z itself is in R, but this contradicts the fact that $z \gamma x \notin R$. Therefore $J \subseteq R$, and thus $J \subseteq P$ and the theorem is proved.

5.15 Theorem : Every Jacobson semisimple Γ- ring is isomorphic to a subdirect sum of right primitive Γ- rings.

Proof : If $J = 0$, then the intersection of the right primitive ideals is P_i is 0. Then by theorem 19 of [12], M is isomorphic to a subdirect sum of Γ- rings, where each $M_i \cong M / P_i$. But each of these is by definition a right primitive Γ- rings.

5.16 Lemma : If $A \neq 0$ is a primitive Γ- ring and A is an ideal of K, then K / A^* is a primitive Γ- ring, where $A^* = \{x \in K : x \gamma A = A \gamma x = 0\}$.

Proof : Let I be a maximal right ideal of A such that it contains no non - zero ideals of A, or $(I : A) = 0$. We can select I so that I is regular and let e be the element of A such that $x - e \gamma x$ is in I for every x of A.

Now I is a right ideal of K, for $I \Gamma K \subseteq A \Gamma K \subseteq A$. Thus $(I \Gamma K) \Gamma (I \Gamma K) \subseteq (I \Gamma K) \Gamma A \subseteq I \Gamma A \subseteq I$. If $I \Gamma K \not\subset I$, then, $I + I \Gamma K = A$, since I is maximal. Thus $e = i + \alpha$, where α is in $I \Gamma K$. Then for any β in $I \Gamma K$, $e \gamma \beta = i \gamma \beta + \alpha \gamma \beta$. Now $e \gamma \beta = \beta + i_1$ for some i_1 in I and $\alpha \gamma \beta$ is in $(I \Gamma K) \Gamma (I \Gamma K) \subseteq I$.

Therefore, $\beta = -i_1 + i \gamma \beta + \alpha \gamma \beta$ is in I, $I \Gamma K \subseteq I$, a contradiction. Thus $I \Gamma K \subseteq I$ and I is a right ideal of K.

Define $I_1 = I + \{x - e\gamma x\}$, where x ranges over K. Then I_1 is a right ideal of K and it is regular, for $x - e\gamma x$ is in I_1 for every x in K. Furthermore $I_1 \neq K$. In particular e is not in I_1, for if e is in I_1, then $e\gamma A \subseteq I_1\gamma A = (I + \{x - e\gamma x\})\gamma A = I\gamma A + (x - e\gamma x)\gamma A \subseteq I$.

Since $a - e\gamma a$ is in I for every a in A, we would then have a in I for every a, $A = I$, a contradiction. Therefore, I_1 is a regular right ideal of K, and $I_1 \neq K$. By zorns lemma we may select a right ideal of K which is maximal with respect to excluding e and including I_1. Let I_2 be this right ideal. Then I_2 is a maximal right ideal of K, for any right ideal of K which properly contains I_2 must contain e. It also contains I_1, and thus must be all of K. This maximal right ideal I_2 is regular, for $x - e\gamma x$ is in $I_1 \subseteq I_2$ for every x of K.

Now $I_2 \cap A \supseteq I$, for $I \subseteq I_1 \subseteq I_2$ and $I \subseteq A$. On the other hand, the element e is not in I_2 and therefore not in $I_2 \cap A$. However e is in A. Then $A \supset I_2 \cap A \supseteq I$. Since I is a maximal right ideal of A, we must have $I_2 \cap A = I$.

Now let us bring A^* into the picture. If $A^* \not\subset I_2$. Then $K = A^* + I_2$, since I_2 is maximal. Now $K\Gamma A = A^*\Gamma A + I_2\Gamma A = I_2\Gamma A$.
But $I_2\Gamma A \subseteq I_2 \cap A = I$. Thus $K\Gamma A \subseteq I$ and in particular, $A\Gamma A \subseteq I$. Thus $A \subseteq (I : A) = 0$, a contradiction. Therefore, $A^* \subseteq I_2$.

We wish to show that K / A^* is primitive, and to do this we shall show that I_2/A^* is a maximal right ideal of K / A^* such that I_2/A^* does not contain any non-zero two sided ideals of K/A^*. It is clear that I_2/A^* is a maximal right ideal of K/A^* because, I_2 is a maximal right ideal of K. To show that 0 is the largest two sided ideal of K/A^* contained in I_2/A^*,

it is sufficient to show that there are no two sided ideals of K between I_2 and A^*. Thus let B be any ideal of K which is contained in I_2.
Then $B \Gamma A \subseteq I_2 \Gamma A \subseteq I$.

Now $B \Gamma A$ is an ideal of A and it is in I . Since I does not contain any non - zero ideals of A , $B \Gamma A = 0$.
Then $(A \Gamma B) \Gamma (A \Gamma B) = 0 = A \Gamma (B \Gamma A) \Gamma B = 0$.
Since A is primitive , it is prime , and thus $A \Gamma B = 0$. Therefore $B \Gamma A = A \Gamma B = 0$, and $B \subseteq A^*$. Thus there are no ideals of K that are contained in I_2 and that properly contains A^* and therefore K/A^* is primitive .

Theorem 5.11 , 5.12 and 5.16 give the following result .

5.17 Theorem : The class of all primitive Γ- rings is a special class of Γ- rings .

5.18 Special radical : A radical is said to be **special** if it is the upper radical determined by the special class of Γ- rings .

5.19 Lemma : The special radical S of any Γ- ring K is equal to the intersection of all ideals T of K such that K / T is a Γ-ring in the special class M . Thus every S - semisimple Γ- ring is a subdirect sum of Γ- rings from M .

The proof is in [30] .

The special radical determined by the class M of all primitive Γ- rings is the intersection of all ideals T such that K / T is a primitive Γ- ring by

Lemma 5.19 .Theorem 5.14 shows that this is specisely the Jacobson radical . Thus we have :

5.20 Theorem : The Jacobson radical is the largest radical for which primitive Γ- rings are semisimple .

5.21 Corollary : The Jacobson radical is a special radical .

Proof : Since Jacobson radical is the upper radical determined by the primitive Γ- rings and the class of all primitive Γ-rings is a special class of Γ- rings . So that the Jacobson radical is a special radical .

CHAPTER – SIX

The radical determined by the maximal ideals of a Γ- ring

In this chapter we have studied a radical which is determined by the maximal ideals of a gamma ring . We characterized this radical by means of the set of regular elements .

6.1 Definition : Let M be a Γ- ring with unity 1 . Then an element $x \in M$ is called **left (or right) regular** in M if there exist elements $y \in M$ and $\delta \in \Gamma$ such that $y \delta x = 1$ ($x \delta y = 1$) .

M is regular if for every element $x \in M$ there exist $y \in M$ and $\delta \in \Gamma$ such that $x \delta y = y \delta x = 1$. If x is not regular then it is singular .

6.2 Maximal ideal : A proper ideal I of a Γ- ring M is said to be **maximal** if there exists an ideal J of M contains I then , either $I = J$ or $J = M$.

Zorns lemma shows that any proper left ideal can be imbedded in a maximal left ideal ; and since the zero ideal $\{0\}$ is a proper left ideal , maximal left ideal certainly exists . We now define the radical $\mathfrak{R}$ of M to be the intersection of all its maximal left ideals . We can write $\mathfrak{R} = \cap L$. $\mathfrak{R}$ is clearly a proper left ideal .

These ideas can be formulated just as easily for right ideals as for left ideals. This means that $\mathfrak{R}$ is also the intersection of all the maximal right ideals R in M, that is $\mathfrak{R} = \cap R$.

6.3 Lemma : If r is an element of $\mathfrak{R}$, then $1 - r$ is left regular.

Proof : Let $(1 - r)$ be left singular.
So that $L = M \delta (1 - r) = \{ x - x \delta r \mid x \in M \}$ is a proper left ideal which contains $1 - r$. We next imbed L in a maximal left ideal M', which contains $1 - r$. Since r is in $\mathfrak{R}$, it is also in M'. Therefore $1 = (1 - r) + r$ is in M'. This implies that $M' = M$, which is a contradiction.

6.4 Lemma : If r is an element of $\mathfrak{R}$, then $1 - r$ is regular.

Proof : By the previous lemma, there exist $s \in M$, $\delta \in \Gamma$ such that $s \delta (1 - r) = 1$. So s is right regular and $s = 1 - (-s) \delta r$. Since $\mathfrak{R}$ is left ideal, $(-s) \delta r$ is in $\mathfrak{R}$. Hence $1 - (-s) \delta r$ is left regular. Since s is both left regular and right regular, it is regular with inverse $1 - r$. So $1 - r$ is also regular.

6.5 Lemma : If r is an element of $\mathfrak{R}$, then $1 - x \delta r$ is regular for every x in M.

Proof : Since $\mathfrak{R}$ is left ideal, so $x \delta r \in \mathfrak{R}$. Therefore, $1 - x \delta r$ is regular by lemma 6.4.

6.6 Lemma : If $r \in M$ with the property that $1 - x \delta r$ is regular for every x, then $R \in \mathfrak{R}$.

Proof : Suppose that r is not in $\Re$. So that r is not in some maximal left ideal M'. It is easy to see that the set $M' + M\,\delta\,\Re = \{ m + x\,\delta\,r \mid m \in M'$ and $x \in M \}$ is a left ideal which contains both M' and r .
So $M' + M\,\delta\,r = M$ and $m + x\,\delta\,r = 1$ for some m and x .
Now $m = 1 - x\,\delta\,r$. Since $1 - x\,\delta\,r$ is regular , m is regular in M' . But this is impossible , for no proper ideal can contain any regular element .

The effect of these Lemmas is to establish the equality of two sets :
$\cap L = \{ r : 1 - x\,\delta\,r$ is regular for every $x \}$ ....(1)
where L is the maximal left ideals.
For the maximal right ideals , we have
$\cap R = \{ r : 1 - r\,\delta\,x$ is regular for every $x \}$ ....(2)

We now prove that all four of these sets are the same by showing that the two sets on the right of (1) and (2) are equal to one another . By symmetry, it evidently suffices to prove the

6.7 Lemma : If $1 - x\,\delta\,r$ is regular , then $1 - r\,\delta\,x$ is also regular .

Proof : We assume that $1 - x\,\delta\,r$ is regular with inverse
$s = (1 - x\,\delta\,r)$. This means that $(1 - x\,\delta\,r)\,\delta\,s = s\,\delta\,(1 - x\,\delta\,r) = 1$

$$\Rightarrow \quad 1\delta\,s - x\,\delta\,r\,\delta\,s = s\,\delta\,1 - s\,\delta\,x\,\delta\,r = 1$$

$$\Rightarrow \quad s - x\,\delta\,r\,\delta\,s = s - s\,\delta\,x\,\delta\,r = 1$$

$$\Rightarrow \quad -x\,\delta\,r\,\delta\,s = -s\,\delta\,x\,\delta\,r = 1 - s$$

$$\Rightarrow \quad -r\,\delta\,x\,\delta\,r\,\delta\,s\,\delta\,x = -r\,\delta\,s\,\delta\,x\,\delta\,r\,\delta\,x = r\,\delta\,(1 - s)\,\delta\,x$$

$$\Rightarrow \quad 1 - r\,\delta\,x\,\delta\,r\,\delta\,s\,\delta\,x = 1 - r\,\delta\,s\,\delta\,x\,\delta\,r\,\delta\,x = r\,\delta\,x - r\,\delta\,s\,\delta\,x + 1 .$$

$$\Rightarrow \quad 1 - r\,\delta\,x + r\,\delta\,s\,\delta\,x \;\; - r\,\delta\,x\,\delta\,r\,\delta\,s\,\delta\,x$$

$$= 1 - r\,\delta\,x + r\,\delta\,s\,\delta\,x - r\,\delta\,s\,\delta\,x\,\delta\,r\,\delta\,x$$

$$= 1$$

$$\Rightarrow (1 - r\,\delta\,x)\,\delta\,(1 + r\,\delta\,s\,\delta\,x) = (1 + r\,\delta\,s\,\delta\,x)\,\delta\,(1 - r\,\delta\,x) = 1\,.$$

So that $1 - r\,\delta\,x$ is regular with inverse $1 + r\,\delta\,s\,\delta\,x$.

We summarize our results in

6.8 Theorem : The radical $\mathfrak{R}$ of M equals to each of the four sets in (1) and (2) and is therefore a proper two sided ideal.

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LIST OF SPECIAL SYMBOLS

M	- Gamma ring
J (M)	– Jacobson radical for gamma ring
P(M)	–Prime radical of M
S(M)	–Strongly nilpotent radical
Z	–Set of integers
R	– Operator ring of the gamma ring
J(R)	– Jacobson radical of the right operator ring
r q r	– Right quasi regular gamma ring
Σ	–Sammation of
$\subseteq$	– Subset of
$\supseteq$	– Superset of
$\in$	– Belong to
$\not\subset$	– Not subset of
$\cap$	– Intersection of
$\cup$	– Union of
$\notin$	– Not belong to
$\wp$	– Prime radical
$\Re$	– Radical class
$\Rightarrow$	–Implies that
$\forall$	– For all
Φ	– Empty set
$\oplus$	– Direct sum
$\cong$	– Isomorphic to

&	– And
< >	– Ideal generated by
etc	– Etcetera
i,e	–That is
$\mathbf{Ann}_M$	-anhilator of M .