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Study of Sums and Products in Some Branches of Mathematics

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Study of Sums and Products in Some Branches of Mathematics

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Certified that the thesis entitled "Study of Sums and Products in Some Branches of Mathematics" submitted by Kalyan Kumar Dey in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi, Bangladesh has been completed under my supervision. I believe that this research work is an original one and that it has not been submitted elsewhere for any degree.

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Kalyan Kumar Dey

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STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

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Synopsis

In many branches of Mathematics a strategy for study of the mathematical entities is to view them as being made up of simpler entities in one or more ways. The task is then divided into two parts : (i) identifying and studying simple entities, (ii) investigating various manners of amalgamating these simple building blocks together. When two mathematical objects of the same nature are glued together to obtain a third such object, the làtter is often called a sum or a product. In this thesis our objects of study are sums and products and their similar counterparts in various areas of Mathematics.

The first chapter gives a survey of various known sums and products in many mathematical branches including Algebra, Topology and Graph Theory.

In the second chapter direct product and wreath products of transformation semigroups have been defined and studies. Their nature and applicability have been investigated and their associavity and mutual distributivity have been established.

In the third chapter two kinds of 'product's of partially ordered sets have been studied. In particular such products of lattices have been considered.

The fourth chapter introduces two kinds of 'sum's for topological spaces. One of them is a generalisation of 'connected sum' of surfaces, while the other is constructed after the pattern of 'amalgamated free product' for groups. Some properties of these products have been studied.

In the fifth and last chapter a number of 'sums' and 'products' have been defined and studied for two objects in a category. These have a maximal or a minimal property in some sense.

Contents

Chapter 1

A Survey of Well-known Sums and Products

Introduction

We shall describe here how in different branches of Mathematics various procedures have been used to amalgamate two or more mathematical objects of the same nature to yield a new such object. The resulting object is usually called a sum or a product. The method of forming such sums and products has often proved very useful for building complex mathematical entities out of a number of simpler objects.

In this chapter we shall give an account of forming sums and products or some such separations in a number of branches of Mathematics and hint at their possible applications. This will illustrate the significance and importance of these processes. The branches of Mathematics considered here include Algebra, Topology and Graph Theory.

1. Algebra

(a) Groups:

(i) External Direct Product

Let A and B be groups. Then the external direct product of A and B, written $A \times B$, is defined as the set $A \times B$ together with multiplication $(a, b).(a', b') = (aa', bb')$. As an example , $C_m \times C_n \cong C_{mn}$, if $(m , n) = 1$ and C_r is the cyclic group of order r.

In general, if $\{A_\alpha\}_{\alpha\in I}$ is any non-empty collection of groups, then the external direct product $\prod_{\alpha\in I} A_\alpha$ is the group is the Cartesian product $\times_{\alpha\in I} A_\alpha$ together with multiplication $\{a_\alpha\}.\{a_{\alpha'}\} = \{a_\alpha a_{\alpha'}\}$.

Sometimes it is called the **unrestricted direct product.**

The restricted direct product is the subgroup S of the external direct product $\coprod_{\alpha\in I} A_\alpha$ consisting of all those elements $\{a_\alpha\}$ for which $a_\alpha \neq e_\alpha$, for only a finite number of α's.

If I is finite, the restricted and the unrestricted direct product coincide.

If $\Re$ denotes the additive group of real numbers, $\Re^n$ is the external direct product of n copies of $\Re$ of with

$\Re^n = \Re\times\Re\times.............\times\Re$ (n copies) and

$(a_1, a_2, ... a_n). (b_1, b_2, ... , b_n) = (a_1+b_1, a_2+b_2, ... a_n+b_n)$.

For abelian groups i.e. commutative groups, written additively, the restricted direct product is called the direct sum (external).

Thus $\Re^n$ is the direct sum of n copies of $(\Re, +)$.

(ii) Internal Direct Product

Let G be a group and let $\{N_\alpha\}_{\alpha\in I}$ be a nonempty collection of normal subgroups of G such that (i) for each $\alpha\in I$, $N_\alpha \cap \bar{N}_\alpha = \{e\}$ where $\bar{N}_\alpha$ is the subgroup of G generated by $\bigcup_{\beta\in I,\beta\neq\alpha} N_\beta$, (ii) G is generated by $\bigcup_{\alpha\in I} N_\alpha$.

In this case, every element of G can be expressed uniquely as a product $n_{\alpha_1} n_{\alpha_2} \ldots\ldots\ldots n_{\alpha_r}$ for some r where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are distinct and the uniqueness is upto the order of n_α's.

Also $n_\alpha n_{\alpha'} = n_{\alpha'} n_\alpha$ for each distinct pair (α, α').

It can be shown that ([], p.34) that the internal direct product $\{N_\alpha\}$ is isomorphic to the restricted external direct product of $\{N_\alpha\}_{\alpha \in I}$.

Direct Sum

If the group is additive abelian, the above product is called the direct sum (internal). The direct sum is denoted by ⊕. Thus

$$Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} \cong \{\bar{0}, \bar{2}, \bar{4}\} \oplus \{\bar{0}, \bar{3}\} = Z_3 \oplus Z_2.$$

Application

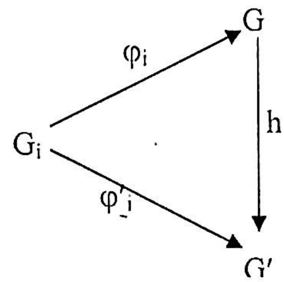
Direct sums are used to describe the structures of certain classes of abelian groups, e.g. free abelian groups, divisible groups and finitely generated abelian groups.

Direct Sum (external and internal) of vector spaces, modules is similarly defined.

We have the following uniqueness proposition about free products.

Proposition ([61], p.97)

Let G and G′ are free products of a collection $\{G_i : i \in I\}$ of groups (with respect to homomorphism $\varphi_i : G_i \to G$ and $\varphi'_i : G_i \to G'$, respectively). Then there exists a unique isomorphism $h : G \to G'$ such that the following diagram is commutative for any $i \in I$:



Theorem ([61], p. 98)

Given any collection $\{G_i: i \in I\}$ of groups, their free products exists.

We denote the free product of groups G_1, G_2, ... G_n by $G_1 * G_2 * \ldots * G_n$ or $\prod_{1 \le i \le n}^{*} G_i$.

The free product of the family of groups $\{G_i: i \in I\}$ is denoted by $\prod_{1 \le i \le n}^{*} G_i$.

Examples

1. A free group on n generators is a free product of n infinite cyclic groups.
2. If $G = \langle x, y \mid x^m = e, y^n = e \rangle$, then G is a free product of cyclic group of order m and n ,i.e., $G = C_m * C_n$.

Application

Free product is used to describe the structure of fundamental groups of path-connected topological spaces.

If X and Y are two path-connected topological spaces such that $X \cap Y$ is a singleton, then $\pi(X \cup Y) = \pi(X) * \pi(Y)$.

(iii) Semidirect Product

Let G be a group, N a normal subgroup and H a subgroup of G such that every element of G can be expressed uniquely a product hn,

$h \in H$, $n \in N$. Then G is called the semidirect product of N by H, denoted by $H \lozenge N$. The element hn is denoted by [h,n], $h \in H$, $n \in N$.

Theorem ([61], p. 89)

If G is the semidirect product of N by H then the elements of [h, 1] of G form a subgroup isomorphic to H and the elements [1, n] form a normal subgroup isomorphic to N. Moreover, the automorphism of N as a subgroup of G is induced by transformation by the element h=[h,1] of H as a subgroup of G. Moreover, $G=H \cup N=HN$.

Theorem ([61], p. 89)

G is the semidirect product of N by H if and only if N is a normal subgroup of G and H is a subgroup of G whose elements may be taken as the coset representation of N. Otherwise expressed

i. N is a normal subgroup of G

ii. H is a subgroup of G

iii. $H \cap N = 1$

iv. $H \cup N = G$.

Examples

S_3 is the semidirect product of C_3 with C_2.

S_3 is the semidirect product of A_3 by $S_2=C_2$.

(iv) Wreath Products of Permutation Groups

Let G and H be permutation groups on sets A and B, respectively. We define **wreath product** of G by H, written GςH in the following way:

GςH is the group of all permutations θ on $A\times B$ of the following kind $\theta(a,b) = (a\Upsilon_b, b\eta)$, $a\in A$, $b\in B$ where for each $b\in B$, Υ_b is a permutation of G on A, but for different b's the choices of the permutations Υ_b are independent. The permutation η is a permutation of H on B. The permutation θ with $\eta=1$ form a normal subgroup G* isomorphic to the direct product of n copies of G, where n is the number of **letters** in the B. The factor group GςH/G* is isomorphic to H, and the permutation θ with all $\Upsilon_b=1$ form a subgroup isomorphic to H, whose elements may be taken as coset representation of G* in G. Let X and Y be finite set with $X\cap Y=0$. Let G be a permutation group on X and H be a permutation group on Y. Then the wreath product GςH is permutation group on $X\times Y$ and the direct product $G\times H$ is permutation group on $X\cup Y$.

(v) Free Product

A group *G* is said to be the **free product** of its subgroups A_α (a ranges over some index set) if the subgroups A_α generate *G*, that is, if every element *g* of *G* is the product of a finite number of the elements of the A_α, $g = a_1a_2 \ldots a_n$, $a_i \in A_i$, $i=1, 2, \ldots, n$, (1)

and if every element *g* of *G*, $g \neq 1$, has a *unique* representation in the form (1) subject to the condition that all the elements ai are different from the unit element and that in (1) no two adjacent elements are in the same subgroup *Aa*-although the product (1) may, in general, contain several factors from one and the same subgroup.

The free product is denoted by the symbol.

$$G = \prod_\alpha^* A_\alpha \qquad (2)$$

and if G is the free product of a finite number of subgroups $A_1, A_2, ..., A_k$, by the symbol

$$G = A_1 * A_2 * ... * A_k.$$

The subgroups A_a are called the **free factors** of the free decomposition (2) of G. The expression (1) (under the restrictions imposed on it) is called the **normal** form (or irreducible representation) of the element g in the decomposition (2), and the number n the length of g in this decomposition; we write $n=l(g)$.

If a group G is generated by subgroups A_a (where a ranges over an index set), then G is the free product of these subgroups if and only if for every group H and every set of homomorphic mappings Φ_a of each A_a into H there exists a homomorphic mapping φ, of G into H that coincides with φ_a on each A_a.

Free Product with an Amalgamated Subgroup

In some connections an even more general construction than that of the free product turns out to be useful. Let A_a be groups, where a ranges over a set of indices, and let a proper subgroup B_α be chosen in every A_α such that all these subgroups are isomorphic to a fixed group B. By φ_α we denote a specific isomorphic mapping of B_α onto B; then $\phi_\alpha\beta = \varphi_\alpha\varphi^{-1}{}_\beta$ is an isomorphic mapping of B_α onto B_β.

The free product of the groups A_α with the amalgamated subgroup B is defined as the factor group G of the free product of the groups A_α with respect to the normal subgroup generated by all elements of the form $b_\alpha b_\beta^{-1}$, *where* $b_\beta = b_\alpha \psi \alpha\beta$, where b_α ranges over the whole subgroup B_α, and where α and β are all possible index pairs. In other words, if every group A_α is given by a system of generators M_α and a system of defining relations Φ_α between these generators, then G has as a system of generators the union of all sets M_α, as a system of defining relations the union of the sets Φ_α and, in addition, all relations obtained by identifying those elements of different subgroups B_α and B_β which are mapped by the isomorphisms φ_α and φ_β onto one and the same element of B. The subgroups B_α are "amalgamated", as it were, in accordance with the isomorphisms $\Psi_{\alpha\beta}$.

We note two important results involving free products.

(α) Kurosh's Subgroup Theorem [19]

$$\text{If } G = \prod\nolimits_\alpha^* A_\alpha \qquad (1)$$

and if H is an arbitrary subgroup of G, then there exists a free decomposition of H.

$$G = F * \prod\nolimits_\alpha^* A_\alpha \qquad (2)$$

where F is a free group and every B is conjugate in G to a subgroup of one of the free factors A_a.

(β) Theorem (Baer and Levi) [19]

No group can be decomposable both into a free product and into a direct product.

Torsion Products of Abelian Groups

For abelian groups A and G we define the torsion product Tor(G, A) as that abelian group which has generators all symbols < g, m, a>, with $m \in Z$, $mg = 0$ in G, and $ma=0$ in A, subject to the relations ("additivity" and "slide" rules for factors m, n)

$$<g_1+g_2, m, a> = <g_1, m, a> + <g_2, m, a>, \quad mg_i = 0 = ma$$

$$<g, m\ a_1+a_2> = <g, m, a_1> + <g, m, a_2>, \quad mg = 0 = ma_i$$

$$<g, mn, a> = <gm, n, a>, \quad mng = 0 = na$$

$$<g, mn, a> = <g, m, na>, \quad mg = 0 = mna$$

Each relation is imposed whenever both sides are defined; in each case this amounts to the requirement that the symbols on the right hand side be defined. The additivity relations imply that $< 0, m, a> =0= <g, m, 0>$. Hence Tor(G, A) = 0 when A has no elements (except 0) of finite order. Also Tor(A, G) $\cong$ Tor(G, A).

In particular, Tor(Z, A) = Tor(Q, A) = Tor(R, A) = 0, for each abelian group A, and $Tor(Z_m, Z_n) \cong Z_{(m, n)}$.

Modules and Vector Spaces

Let R be a commutative ring with 1. Let A and B be two R-modules. The **tensor product** of A and B, written $A \otimes_R B$ is the R-module generated by all symbols $a \otimes b$, $a \in A$, $b \in B$ with defining relations.

(i) $(a_1+a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$

(ii) $a \otimes (b_1+b_2) = a \otimes b_1 + a \otimes b_2$

(iii) $ar \otimes b = a \otimes rb$,

for each $a_1, a_2, a \in A$, $b_1, b_2, b \in B$, $r \in R$.

As particular cases,

$R \otimes_R A \cong A$, $Z \otimes_Z Z_m \cong Z_m$, $Z_m \otimes Z_n \cong Z_{(m, n)}$.

The tensor product of two vector spaces U and V over a field F is defined similarly. If dim U=m and dim V=n, then dim $(U\otimes_F V) = mn$.

Tensor product of modules and vector spaces are associative in the sense:

$$\begin{cases}(A\otimes_R B)\otimes_R C \cong A\otimes_R (B\otimes_R C),\\ (U\otimes_F V)\otimes_F W \cong U\otimes_F (V\otimes_F W).\end{cases}$$

Also tensor product is distributive over direct sum in the following sense:

$$\begin{cases}A\otimes_R(B\oplus C) \cong (A\otimes_R B) \oplus (A\otimes_R C),\\ (A\oplus B) \otimes_R C \cong (A\otimes_R C) \oplus (B\otimes_R C).\end{cases}$$

$$\begin{cases}U\otimes_F(V\oplus W) \cong (U\otimes_F V) \oplus(U\otimes_F W),\\ (U\oplus V) \otimes_F W \cong (U\otimes_F W) \oplus (V\otimes_F W).\end{cases}$$

Cayley-Dickson Construction of Division Algebras

An **algebra** A is a vector space equipped with a multiplication of vectors and a multiplicative identity. A **division algebra** is an algebra A such that if a, b $\in$A and ab=0 then either a = 0 or b=0, i.e., every non-zero element of A has a multiplicative inverse.

There are only 4 normed division algebras: **R**, the real numbers, C, the complex number, H, the quaternions, and O, the octonions. C= has a basis {1,i} over R with i^2 =-1.H has a basis {1, i, j, k} over R with $i^2=j^2=k^2$ = ijk=-1.O has a basis $\{1,e_1,e_2,\ldots, e_7\}$ over R with

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_4	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

There is an interesting method of constructing these algebras step by step.

A algebra is an algebra equipped with a conjugation, i.e., real linear map $*: A \to A$ with $a^{**} = a$, $(ab)^* = b^* a^*$.

A complex number can be thought of as an order (a, b) or real numbers with component-wise addition and multiplication given by $(a, b)(c, d) = (ac-db, ad+cb)$.

We can also define conjugate of a complex number by $(a, b)^* = (a, -b)$.

We can then define a quaternion in a similar way. A quaternion is a pair of complex numbers. Addition is component-wise and multiplication is given by $(a, b)\cdot(c, d) = (ac-db^*, a^*d + cb)$ (2)

We can also define the conjugate of a quaternion by (3) $(a, b)^* = (a^*, -b)$.

We can now define an octonion by an ordered pair of quaternions. As before we can define addition and multiplication using (2) and (3).

This method can be continued to obtain a new algebra from an old one. This method is called **Cayley-Dickson construction.**

Let G and H be two groups given by the following generators and relations: $G = \langle x_1 \ldots; r_1 \ldots \rangle$ and $H = \langle y_1 \ldots; s_1 \ldots \rangle$ Let A and B be subgroups of G and H respectively such that their exists an isomorphism $\varphi : A \rightarrow B$. Then the free product of G and H, amalgamating the subgroups A and B by their isomorphism φ is the group $\langle x_1, \ldots ; y_1, \ldots ; r_1, \ldots , s_1, \ldots ; a = \varphi(a), a \in A \rangle$.

An important concept very closely related to the free product of two groups with an amalgamated subgroup is the Higmann-Neumann-Neuman extension or HNN extension of a group relative to two of its subgroups A, B and an isomorphism $\varphi : A \rightarrow B$.

HNN Extension

Let G be a group and Let A and B be subgroups of G with $\varphi : A \rightarrow B$ an isomorphism. **The HNN extension of G related to A, B and φ is the group $G^* = \langle G, t; t^{-1}at = \varphi(a), a \in A \rangle$.**

Free products with amalgamation and HNN extensions are often motivated by consideration of fundamental groups of topological spaces.

All topological space are assumed to be path connected. If X is a topological space π (X) will be denote the fundamental groups of X. Let X and Y be space, and Let U and V be open path connected subspaces of X and Y respectively such that there is a homeomorphism, $h: U \rightarrow V$. Choose. Choose a base point $\mu \in U$ for the fundamental groups of U and X. Similarly, choose as base point $h(\mu) = v \in V$. There is a homomorphism $\eta: \pi_1(U) \rightarrow \pi_1(X)$ defined by simply considering a loop in U as a loop in X. Suppose that η and the similarly defined hommomrphism $\delta : \pi_1(V) \rightarrow \pi_1(Y)$ are both injections. The homeomorphism h induces an isomorphism $h^*:\pi_1(U)\rightarrow \pi_1(V)$. Suppose we identify U and V by the homeomorphism h to obtain a new space Z. Under the assumptions made, the Seifert-van Kampen Theorem (Massey [61]) says that

$$\pi_1(Z) = (\pi_1(X)* \pi_1(Y); \pi_1(V), h^*>$$

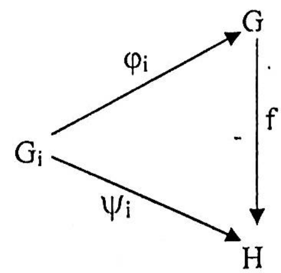
The HNN extension has a similar topological interpretation. Suppose that U and V are both subspaces of the arcwise connected space X. Assume the same hypothesis on U and V as above. Let I be the unit interval, and let C=U×I. Identify $U \times\{0\}$ with U and identify $U\times\{1\}$ with V by the homeomorphism h. Let Z be the resulting space. (What we have done is to attach a handle to X.) The Seifert-van Kampen Theorem can be used to show that $\pi_1(Z) = < \pi_1(X),t ;t^{-1}ut = h(u),u \in U >$

Free Products of Groups

Let $\{G_i\}_{i\in I}$ be a collection of groups, and assume there is given for each index i a homomorphism φ_i of G_i into a fixed group G. We say that G is the **free product** of the groups G_i (with respect to the homomorphism φ_i) if and only if the following condition holds:

For any group H and any homomorphism $\psi_i : G_i \rightarrow H$, $i\in I$

there exists a unique homomorphisms f: G→H such that for any $i\in I$, the following diagram is commutative



(b) Rings :

(i) Complete Direct Sum and Subdirect Sum

Let $\{A_i : i \in I\}$ be a class of rings.

Consider the set S of symbols $(a_1, a_2, ...)$ where a_i belongs to A_i. These are infinite vectors. Addition and multiplication are defined co-ordinatewise.

$$(a_1, a_2, ...) + (b_1, b_2, ...) = (a_1+b_1, a_2+b_2, ...)$$

$$(a_1, a_2, ...). (b_1, b_2, ...) = (a_1b_1, a_2b_2, ...)$$

Then S is a ring and if is called the **complete direct sum** of the rings A_i.

The set of all elements of the form $(0, 0, ... 0, a_i, 0, ...)$ is a subring A'_i which is isomorphic to A_i and the mapping

$(a_1, a_2, ... , a_i, ...) \rightarrow (0, 0, ... 0, a_i, 0, ...)$ is homomorphism of S onto A'_i. The subring which consists of all elements of S which have only a finite number of nonzero entries is called the **weak direct sum** of the A_i. The weak direct sum is also called the **direct sum**.

Thus the natural homomorphisms of the weak direct sum to A'_i are onto for every i. We say that a subring S^* of the complete direct sum S is a **subdirect sum** of the rings A_i if natural homomorphism of S^* to A'_i

$(a_1, a_2, ... , a_i, ...) \rightarrow (0, 0, ... 0, a_i, 0, ...)$ is an onto mapping for every i .

The subdirect sum has a very useful characterisation in the following result.

Theorem ([34],p 64)

A ring R is isomorphic to a subdirect sum of the rings A_i if and only if R contains a class of ideals $\{B_i\}$ such that $\bigcap_i B_i = 0$ and $R/B_i \cong A_i$.

(c) Radicals

A non-empty class R or rings is called a radical or a radical class if

(α) R is homomorphically closed;

(β) every ring A has an *R*-ideal R(A) which contains every *R*-ideal of A;

(γ) for every ring A, $\frac{A}{R(A)}$ has no non-zero *R*-ideals.

If R(A) = A, A is called an ***R*-radical** ring and if R(A) = 0, A is called an ***R*-semisimple** ring.

Examples

(1) The set of nil rings is a radical *N*. Here a nil ring is a ring. A such that for each x is A, $x^n=0$ for some positive integer n.

(2) The set of all right quasi-regular rings is a radical J. A ring A is called *right quasi regular* if for each x∈A, there is an element y∈A such that $x + y + xy = 0$

Given a non-empty class of rings C, there is a smallest radical containing C. It is called the *lower radical determined by C.* and is denoted by L_C. If C satisfies the condition: *every non-zero ideal of a non-zero ring in C can be mapped homomorphically onto a non-zero ring in C*, then the class U_C of rings A which cannot be mapped onto a non-zero ring in C is radical and is the largest radical with respect to which A is semisimple.

We now describe a number of methods of forming a new radicals out of two given radicals.

(i) Join (Sum) and Meet (Product) of two Radicals

Let R_1 and R_2 be two radicals, then the **Join** or the **sum** of R_1 and R_2 is the radical $L_{R_1 \cup R_2}$ (see Leavitt [60], Majumdar and Dey [49].

For two radicals R_1 and R_2, the meet or **the product** of R_1 and R_2 is the radical $R_1 \cap R_2$ ([55], [43]). This is denoted by $R_1 \wedge R_2$ or $R_1 R_2$ and is the largest radical contained in both R_1 and R_2.

(ii) Radical Pairs and their likes

For two radicals R_1 and R_2 Snider [43] defined a radical $(R_1 : R_2)$, called the **radical pair,** defined as

(1) $(R_1 : R_2)$ = The class of all rings A such that for each ideal I of A, $R_1 (\frac{A}{I}) \supseteq R_2 (\frac{A}{I})$.

Majumdar and Paul [46] generalised this construction and obtained three radicals and called them **radicals similar to radical pairs.** These are :

(2) $(R_1 ; R_2)$ = The class of all rings A such that for each ideal I of A and each ideal J of I, $R_1 (\frac{I}{J}) \supseteq R_2 (\frac{I}{J})$.

(3) $(R_1 \mid\mid R_2)$ = The class of all rings A such that for each ideal I of A, $R_1 (\frac{A}{I}) = R_2 (\frac{A}{I})$.

(4) $(R_1 \ominus R_2)$ = The class of all rings A such that for each ideal I of A and for each ideal J of I, $R_1(\frac{I}{J}) = R_2(\frac{I}{J})$.

The classes of rings (1) –(4) are not always radicals. Conditions for which these classes of rings are radicals have been stated and proved in [43] and [46].

(d) Lattices

Majumdar and Sultana [ ] defined two kinds of internal and external chain products and also internal and external tag products of lattices. Bae and Lee [ ] define truncated product and Bennet [ ] defined rectangular product of lattices.

We describe :

There are a number of products of lattices. We shall describe here some of them.

(i) Direct Product

Let L and K be lattices. Consider L×K, the Cartesian product of the sets L and K. Define two binary operations ∧ and ∨ on L×K by

$(a, b) \wedge (a', b') = (a \wedge a', b \wedge b')$,

$(a, b) \vee (a', b') = (a \vee a', b \vee b')$.

Then L × K becomes a lattice under these operations. This lattice is called the direct product of L and K. The definition of direct product can be extended to an arbitrary family of lattices.

Example

The set of all real numbers is a lattice under the usual ordering. If for $z = a + ib$, $z' = a' + ib'$ ($a, b, a', b' \in \Re$), we define $z \leq z'$ if either $a \leq a'$, or ($a = a'$ and $b \leq b'$) then **C**, the set or complex numbers is a lattice under this ordering and is isomorphic to the direct product $\Re \times \Re$.

(ii) Free Distributive Product [13]

Let L_1 and L_2 be two disjoint distributive lattices. Then $Q = L_1 \cup L_2$ is a partially ordered set. A free lattice generated by Q over the class D of all distributive lattices is called a free distributive product L_1 and L_2.

(iii) Tensor Product of Distributive Lattices

Let A, B and C be distributive lattices. A Function $f : A \times B \rightarrow C$ is called a bihomomorphism if the functions $g_a : B \rightarrow C$ defined by $g_a(b) = f(a, b)$ and $h_b : A \rightarrow C$ defined by $h_b(a) = f(a, b)$ are lattice homomorphism for each $a \in A$ and $b \in B$.

Let A and B be distributive lattices. A distributive lattices C is a tensor product of A and B if there exists a canonical bihomomorphism $f : A \times B \rightarrow C$ such that C is generated by $f(A \times B)$ and for any distributive and any bihomomorphism $g : A \times B \rightarrow D$, there is a homomorphism $h : C \rightarrow D$ satisfying $g = hf$.

(i) Internal Chain Products [55]

Let $(L, \leq)$ be a lattice and A, B be two sublattices of L such that $A \not\subset B$, $B \not\subset A$.

$A \cup B$ will be called the **internal chain product** (first kind) of A with B if

(i) $A \cap B = \varnothing$,

(ii) for each $x \in A$ and $y \in B$, $y \leq x$.

We denote this product by A.B. Clearly $A \cdot B$ is a sublattice of L.

$A \cup B$ will be called the **internal chain product** (second kind) of A with B if

(i) $A \cap B = C$ (say), $C \neq \phi$,

(ii) for each $x \in A - B$, $y \in C$ and $z \in B - A$, $z \leq y \leq x$.

We denote this product by $A \odot B$. Then $A \odot B$ is a sublattice of L.

(ii) External Chain products [55]

Let $(A, \leq_1)$ and $(B, \leq_2)$ be two disjoint lattices. Let $X = A \cup B$ and define a relation $\leq$ on X by

(i) for each $x, y \in A$, $x \leq y$ if $x \leq_1 y$,

(ii) for each $x, y \in B$, $x \leq y$ if $x \leq_2 y$,

(iii) for each $x \in B$, $y \in A$, $x \leq y$.

Then $\leq$ is a partial ordering of X and $(X, \leq)$ is a lattice. We call it the external chain product (First kind) of A with B and denoted by $A * B$.

Let $(A, \leq_1)$ and $(B, \leq_2)$ be two disjoint lattices such that there exists sublattices C of A and D of B with the properties:

(i) for each $x \in A - C$ and for each $y \in C$, $y \leq_1 x$;

(ii) for each $u \in B - D$ and for each $v \in D$, $u \leq_2 v$;

(iii) there is an isomorphism f from C onto D.

Define an equivalence relation ~ on $A \cup B$ by the partition

$\Pi = \{\{x\} \mid x \in (A - C) \cup (B - D)\} \cup \{\{x, f(x)\} \mid x \in C\}$.

Thus, for each $x \in (A - C) \cup (B - D)$, cls x = {x}, and for each x $\in C$, cls x = cls f(x) = {x, f(x)}. Let $X = \frac{A \cup B}{\sim}$. Define a relation $\leq$ on X by

(a) for each x, y $\in$ A, cls x $\leq$ cls y if $x \leq_1 y$,

(b) for each x, y $\in$ B, cls x $\leq$ cls y if $x \leq_2 y$,

(c) for each $x \in B - D$ and for each $y \in A - C$, cls x $\leq$ cls y.

Then $\leq$ is a partial ordering on X and (X, $\leq$) is a lattice.

We call X the **external chain product** (second kind) of A and B, and denote it by AΘB.

(iv) Internal Tag Product [55]

Let (L ,$\leq_L$) be a lattice and let A and B be two distinct bounded sublattices of L such that

(1) $A \cap B = \{1_A, 1_B, 0_A, 0_B\}$

(2) $1_A = 1_B$, $0_A = 0_B$.

Define the relation $\leq$ on $A \cup B$ by the following:

(i) for each $a_1, a_2 \in A$, $a_1 \leq a_2$ if $a_1 \leq_L a_2$,

(ii) for each $b_1, b_2 \in A$, $b_1 \leq b_2$ if $b_1 \leq_L b_2$

(iii) for each $a \in A$, $b \in B$, $a \neq 1_A, 0_A$ and $b \neq 1_B, 0_B \Rightarrow a \not\leq b$, $b \leq a$.

Then $\leq$ is a partial ordering and $A \cup B$ is a lattice under $\leq$. This will be called the **internal tag product** of A and B and will be denoted by AoB. Clearly, AoB is bounded, and $1_{AoB} = 1_A = 1_B$, $0_{AoB} = 0_A = 0_B$.

(v) External Tag Product [55]

Let $(A, \leq_1)$ and $(B, \leq_2)$ be two disjoint bounded lattices. Let $X = A \cup B$, and let $\sim$ be the equivalence relation of X defined by the partition: $\Pi = \{\{x\} \mid x \neq 1_A, 1_B, 0_A, 0_B\} \cup \{\{1_A, 1_B\}, \{0_A, 0_B\}\}$. Thus, cls 1_A = cls 1_B $= \{1_A, 1_B\}$, cls 0_A = cls 0_B $= \{0_A, 0_B\}$, cls $x = \{x\}$, $x \neq 1_A, 1_B, 0_A, 0_B$.

Define a relation $\leq$ on $\frac{X}{\sim}$ by the following:

(i) for each x, y $\in$A, cls x $\leq$ cls y $\leq$ if $x \leq_1 y$

(ii) for each x, y $\in$B, cls x $\leq$ cls y if $x \leq_2 y$

(iii) for each $x \in A$, $y \in B$, $x \neq 1_A, 0_A$, and $y \neq 1_B, 0_B$, $\Rightarrow$ cls x $\leq$ cls y, cls y $\leq$ cls x.

Then $\leq$ is a partial ordering and $\frac{X}{\sim}$ is a lattice under $\leq$. It will be called the external tag product of A and B.

We denote it by $A \square B$.

All lattices L with 1 to 6 elements are expressed in terms of chain products and tag products as follow:

Number of elements	Lattices
1	S^1
2	S^2
3	S^3
4	S^4, $S^3 \square S^3$
5	S^5, $S^1.(S^3 \square S^3)$, $(S^3 \square S^3).S^1$, $S^3 \square S^3 \square S^3$, $S^4 \square S^3$, S^6, $S^1.(S^4 \square S^3)$, $(S^4 \square S^3).S^1$, $S^1.(S^3 \square S^3 \square S^3)$, $(S^3 \square S^3 \square S^3).S^1$
[illegible]	[illegible]

Majumdar and Sultana [55] defined two kinds of internal and external chain products and also internal and external tag products of lattices. Bae and Lee [63] define truncated product and Bennet [64] defined rectangular product of lattices.

We describe :

There are a number of products of lattices. We shall describe here some of them.

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Let L and K be lattices. Consider L×K, the Cartesian product of the sets L and K. Define two binary operations $\wedge$ and $\vee$ on L×K by

$$(a , b) \wedge (a' , b') = (a \wedge a' , b \wedge b'),$$

$$(a , b) \vee (a' , b') = (a \vee a' , b \vee b').$$

Then L × K becomes a lattice under these operations. This lattice is called the direct product of L and K. The definition of direct product can be extended to an arbitrary family of lattices.

Example

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Let L_1 and L_2 be two disjoint distributive lattices. Then $Q = L_1 \cup L_2$ is a partially ordered set. A free lattice generated by Q over the class D [illegible]

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Let A and B be distributive lattices. A distributive lattices C is a tensor product of A and B if there exists a canonical bihomomorphism $f : A \times B \to C$ such that C is generated by $f(A \times B)$ and for any distributive and any bihomomorphism $g : A \times B \to D$, there is a homomorphism $h : C \to D$ satisfying $g = hf$.

2. Topology

(i) **Product space**

Let X and Y be two topological space. Consider the Cartesian product $X \times Y$ together is with the topology **T** of generated by all sets of the form $U \times Y$ and $X \times V$, where U and V are open sets in X and Y respectively. This topological space $X \times Y$ is called the **product of X with Y.**

Example

The Euclidean plane with the usual metric topology is the product of the real line with itself.

(ii) Sum

Let X and Y be two topology spaces such either $X \cap Y = \phi$ or the topology on $X \cap Y$ as subspaces of X and Y are identical. Then $X \cup Y$

together with the topology **T** consisting of all subsets of $X \cup Y$ which are of the form $G \cup H$, where G and H are open in X and Y respectively, has been defined as the sum of X and Y and has been denoted by X+Y in [ ]. If $X \cap Y = \phi$, the sum is called the **direct sum** and X+Y is written $X \oplus Y$

.

(iii) Fibre Space

Let X, B be a topological space, a map p: X →B is called (Serre) fibre map it for each polyhedron $|K|$, map f_o: $|K| \to X$ and homotopy $g_t : |K| \to B$ with $g_0 = f_0 p$ there is a homotopy $f_0 : |K| \to X$ with $g_t = f_t p$. If we pick a base point $b_o \in B$ then $p^{-1}b_0$ called the fibre (over b_0), X is called the **fibre space** or **total space** and B the base-space of the fibration.

Connected Sum

Let S_1 and S_2 be disjoint surfaces .The **connected sum,** denoted by $S_1\#S_2$, is formed by cutting a small circular hole in each surface, and then gluing the two surfaces together along the boundaries of the holes. We choose the subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ such that D_1 and D_2 are closed disc (i.e., homeomorphic to E^2). Let S'_i denote the complement of the interior of D_i in S_i for i = 1 and 2. Choose a homeomorphism h of the boundary circle of D_1 onto the boundary of D_2. Then $S_1\#S_2$ is the quotient space of $S'_1 \cup S'_2$ obtained by identifying the points x and h(x) for all points x in the boundary of D_1. It is clear that $S_1\#S_2$ is a surface.

(iv) Suspension

For a space *X*, the suspension *SX* is the quotient of $X \times I$ obtained by collapsing $X\times\{0\}$ to one point and $X\times\{1\}$ to another point. The motivating example is $X = S^n$, when $SX = S^{n+1}$ with the two 'suspension points' at the north and south poles of S^{n+1}, the points (0,...,0, ±1). One can regard SX as a double cone on X, the union of two copies of the cone $CX=(X \times I)/(X \times \{0\})$. If X is a CW complex, so are SX and CX as quotients of $X \times I$ with its product cell structure, I being given the standard cell structure of two 0-cells joined by a 1-cell.

(v) Join

The cone CX is the union of all line segments joining points of X to an external vertex, and similarly the suspension SX is the union of all line segments joining points of X to two external vertices. More generally, given X and a second space Y, one can define the space of all lines segments joining points in X to points in Y. This is the join X*Y, the quotient space of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. Thus we are collapsing the subspace $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y.

(vi) Wedge Sum

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the **wedge sum X ∨ Y** is the quotient of the disjoint union $X \cup Y$ obtained by identifying x_0 and y_0 to a single point. For example, $S^1 \vee S^2$ is homeomorphic to the figure '8', two circles touching at a point. More generally one could form the wedge sum $\angle_\alpha X_\alpha$ of an arbitrary collection of spaces X_α by starting with the disjoint union $\Pi_\alpha X_\alpha$ and identifying points $x_\alpha \equiv X_\alpha$ to a single point. In case the spaces X_α are cell complexes and the points x_α are 0-cells, then $\vee_\alpha X_\alpha$ is a cell complexes since it is obtained from the cell complex $\Pi_\alpha X_\alpha$ by collapsing a subcomplex to a point.

(vii) Smash Product

Inside a product space $X \times Y$ there are copies of X and Y; namely $X\times\{y_0\}$ and $\{x_0\}\times Y$ for points $x_0 \in X$ and $y_0 \in Y$. These two copies of X and Y in $X\times Y$ intersect only at the point (x_0, y_0), so their union can be identified with wedge sum $X \vee Y$. The **smash product** $X \wedge Y$ is then defined to be the quotient $X \times Y / X \vee Y$. One can think of $X \wedge Y$ as a reduced version of $X \times Y$ obtained by collapsing away the parts that are not genuinely a product, the separate factors X and Y.

(vii) CW- complex [1]

An important and useful method of building up complex topological out of simple one is provided by the following procedure. The constituents are called n-cells and the constituted structure is called a **cell complex** or **CW-complex.**

We describe below the procedure :

(1) Start with a discrete set X^0 ,whose points are regarded as 0-cells.

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \cup_\alpha e^n_\alpha$ of $X^{n-1}$with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus a set $X^n = X^{n-1} \cup_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.

(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology : A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

3. Graphs

Let G_1 and G_2 be graphs with disjoint points sets V_1 and V_2 and live sets X_1 and X_2 respectively.

(i) Union

The union $G=G_1 \cup G_2$ is a graph with $V=V_1 \cup V_2$ and $X=X_1 \cup X_2$.

(iii) Join

The join of G_1 and G_2 is G_1+G_2 is the graph of consisting $G_1 \cup G_2$ and all lines joining V_1 with V_2.

(iii) Product

The product $G_1 \times G_2$ has $V=V_1 \times V_2$ and any two pts $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adj v_2) or ($u_2 = v_2$ and u_1 adj v_1).

(iv) Composition

The composition $G = G_1 [G_2]$ has $V=V_1 \times V_2$ and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ wherever (u_1 adj v_1 or $u_1 = v_1$ and u_2 adj v_2).

4. Categories

A category is a class **A**, together with a class M which is a disjoint union of the form $M = \bigcup_{A,B \in A} [A,B]_A$ (To avoid logical difficulties we postulate that each $[A, B]_A$ is a set). For each triple A, B, C $\in$**A**, we are to have a function from [B, C] × [A, B] into [A, C]. The image of the pair (β, α) under this function will be called the composition of β by α, and will be denoted by βα. The composition function are subject to axioms.

(i) Associatively : When ever the compositions make sense are have $(\gamma\beta)\alpha = \gamma(\beta\alpha)$.

(ii) Existence of identities : For each A $\in$**A**, we have an element $1_A \in$ [A, A] such that $1_A\alpha = \alpha$ and $\beta 1_A = \beta$ whenever the compositions make sense.

The members of A are called objects and the members of M are called morphisms. If $\alpha \in [A, B]$ we shall call A the domains and B the codomain of α. α is called a morphism from A to B and this is represented by '$\alpha : A \rightarrow B$' or '$A \xrightarrow{\alpha} B$'.

If **A** is a set the category is called small.

The Nonobjective Approach

A category can also be defined as a class M, together with a binary operation on M, called composition, which is not always defined (that is a function from a subclass of M×M to M). The image of the pair (β,α) under this operation is denoted by βα (if defined). An element $e \in M$ is called an identity if $e\alpha=\alpha$ and $\beta e=\beta$ whenever the compositions make sense. We assume the following axioms:

(i) If either $(\gamma\beta)\alpha$ or $\gamma(\beta\alpha)$ is defined, the other is defined, and they are equal.

(ii) If $\gamma\beta$ and $\beta\alpha$ are defined and β is an identity, then $\gamma\beta$ is defined.

(iii) Given $\alpha \in M$, there are identities and e_R in M such that $e_L\alpha$ and αe_R are defined (and hence equal α).

(iv) For any pair of identities e_L and e_R, the class $\{\alpha \in M \mid (e_L\alpha)e_R$ is defined$\}$ is a set.

Examples

1. The category S whose class of objects is the class of all sets, where $[A, B]_S$ is the class of all functions from A to B, is called the category of sets. **S** is not small.

2. A similar definition applies to the category **T** of all topological spaces where the morphisms from space A to space B are the continuous functions from A to B.

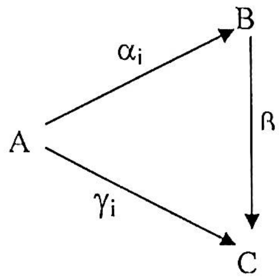
Special Morphisms

$\theta : A \rightarrow B$ is a coretraction if there exists $\theta' : B \rightarrow A$ such that $\theta'\theta = 1_A$. A is called a retract of A. Dually, θ is a retraction if there is $\theta'' : B \rightarrow A$ such that, $\theta\theta'' = 1_B$. If θ is both a retraction and coretraction then θ is called an isomorphism or equivalence. If $\alpha : A' \rightarrow A$ is a monomorphism, A′ is called a **subobject** of A.

Dually if $\alpha : A \rightarrow A'$ is an epimorphism, A′ is called a quotient object of A.

Equivalence

A diagram



is called **commutative** if $\beta\alpha = \gamma$ and we say in this case that γ **factors through** β.

Pull backs and Push out

Given two morphisms $\alpha 1 : A_1 \rightarrow A$ and $\alpha_2 : A_2 \rightarrow A$ with a common codomain, a commutative diagram is called a pull back for $\alpha_1\, \alpha_2$ if for every pair of morphisms $\beta_1' : P \rightarrow A$, $\beta_2' : P \rightarrow A_2$ such that $\alpha_1 \beta_1' = \alpha_2 \beta_2'$ there exists a morphism $\gamma : P' \rightarrow P$ such that $\beta_1' = \beta_1\gamma$, $\beta_2' = \beta_2\gamma$.

Intersections

If the intersection exists for every set of subobjects of every object in **A** we say that **A** has intersections. If intersection exists only for every finite set of subobjects we say that **A** has a finite intersections.

Proposition

If $A_1 \rightarrow A$ and $A_2 \rightarrow A$ are monomorphisms in a category **A**, then the diagram $P \rightarrow A_2$ is a pullback if and only if

$P \rightarrow A_2 \rightarrow A = P \rightarrow A_1 \rightarrow A$ is the intersection of A_1 and A_2. Hence if A has pull backs then A has finite intersections.

The dual of intersection is cointersection

Product and Coproduct

Let $\{A_i\}_{i\in I}$ be a set of objects in an arbitrary category A. A **product** for the family is a of morphisms $\{p_i : A \to A_i\}_{i\in I}$ with the property that for any family $\{\alpha_i : A' \to A_i\}_{i\in I}$ there is a unique morphism $\alpha : A' \to A$ such that $p_i\alpha = \alpha_i$,for all $i\in I$. The object A is denoted by $\underset{i\in I}{\times} A_i$. The morphisms p_i are called the projection morphism from the product. Each p_i is a retraction. In particular, this will always be true when A has a zero object. In this case we can define morphisms $u_i : A_i \to A$ is called the injection morphisms into the product such that $p_j u_i \to \delta_{ij}$ where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1_A$.

The **coproduct** of the family $\{A_i\}_{i\in I}$ is defined dually to the product .Thus the coproduct is a family of morphisms $\{u_i : A_i \to A\}$ called injections such that for each family of morphisms $\{\alpha_i : A_i \to A'\}_{i\in I}$ we have a unique morphisms $\alpha : A \to A'$ with $\alpha u_i = \alpha_i$,for all $i \in I$. The object A is denotes by $\underset{i\in I}{\oplus} A_i$. If A has a zero object then we can define projections $p_i : A \to A_i$ such that $p_j u_i = \delta_{ij}$.

Suppose that for all $i\in I$ we have $A_i = A$. In this case we denote the product of the family by A^I and the coproduct by ${}^I A$. We have the diagonal morphism $\Delta : A \to A^I$ defined by $P_i\,\Delta = 1_A$ for all $i\in I$, and dually, the **codiagonal** morphism $\nabla : {}^I A \to A$ defined by $\nabla ui = 1A$ for all $i\in I$. Then Δ is necessarily a monomorphism and ∇ is necessarily an epimorphism.

In a general category A, a morphism f from the coproduct $\underset{j\in I}{\oplus} A_j$ to a product $\underset{i\in I}{X} B_i$ is completely determined by its coordinate morphisms

$f_{ij}=P_i f u_j$, where μ_j is the j-th injection into the coproduct and P_i is the ith projection from the product.

For all $A' \in A$ the set of morphisms [A′, A] is in 1-1 correspondence with the cartesiam product of sets $\underset{i \in I}{X}[A', A_i]$ $\left[\bigoplus_{j \in J} A_j, \underset{i \in I}{X} B_i\right]$ is in 1-1 correspondence with the set of all I×J matrices of the form (f_{ij}) where $f_{ij} \in [A_j, B_i]$. We shall frequently denote such a morphism by its corresponding matrix.

In particular, when A has a zero object, we have the morphism $\delta=(\delta_{ij})$: $\left[\bigoplus_{j \in I} A_i \to \underset{i \in I}{X} A_i\right]$. If δ is an isomorphism then $\left[\bigoplus_{j \in I} A_i\right]$ is called a **biproduct.**

Chapter 2

Direct Product and Wreath Product of Transformation Semigroups

Introduction

Here we have defined and studied two kinds of amalgamating two transformation semigroups to obtain a third transformation semigroup. These have been called direct product and wreath product and have been defined as generalisations of the corresponding concepts for transformation groups. Uses of such products have been illustrated, and the associability of both the products as well as the distributivity of wreath product over direct product have been established.

Definition

Let S be a semigroup and X a non-empty set. S will be called a **transformation semigroup** on X if there is a mapping ϕ: $S \times X \rightarrow X$, for which we write $\phi(s, x) = s(x)$ and which satisfies the condition $(s_1 s_2)(x) = s_2(s_1(x))$, for each $x \in X$ and for each $s_1, s_2 \in S$.

If S is a monoid, i.e., if S has an identify element 1, then the mapping ϕ is further assumed to satisfy $1(x) = x$, for each $x \in X$.

Clearly, the semigroup E(X) of all endomappings of X is a transformation semigroup on X. For every transformation semigroup S on X, is a homomorphism ψ: $S \rightarrow E(X)$ given by $\psi(s) = f$, where $f(x) = s(x)$. If $\psi(s_1) = f_1$, $\psi(s_2) = f_2$, then $(s_1 s_2)(x) = s_1(s_2(x)) = f_1(f_2(x)) = (f_1 f_2)(x)$, and so, $\psi(s_1 s_2) = f_1 f_2$. Thus, ψ is indeed a homomorphism.

E(X) will be called the **full transformation semigroup** on X. If X is finite, so is E(X); and if the number of elements of X is n, the number of elements of E(X) is n^n. If S is a semigroup, then End S , the set of all endomorphisms of S, i.e. the set of all homomorphisms of S into itself, is a subsemigroup of E(S) under the composition of maps.

Direct Product and Wreath Product of Transformation Semigroups

Direct product and wreath product of transformation groups are well known and useful concepts (see [ ]). Using Majumdar's ideas we generalize these concepts of transformation semigroups and give some characterizations and study their properties.

We consider two transformation semigroups S_1 and S_2 on non-empty sets X_1 and X_2 respectively, we shall see how S_1 and S_2 naturally yield transformation semigroups on $X_1 \cup X_2$ and $X_1 \times X_2$ respectively.

Definition

The **direct product** of S_1 and S_2, written $S_1 \times S_2$, is defined as a transformation semigroup on $X_1 \cup X_2$, the elements of $S_1 \times S_2$ being the ordered pairs (s_1, s_2), $s_1 \in S_1$, $s_2 \in S_2$, with $(s_1, s_2)(x_1) = s_1(x_1)$,$(s_1, s_2)(x_2) = s_2(x_2)$, for each $x_1 \in X_1$, $x_2 \in X_2$. The multiplication in $S_1 \times S_2$ is component-wise. It is easily seen that $S_1 \times S_2$ is indeed a transformation semigroup. It S_1 and S_2 are finite, the number of elements of $S_1 \times S_2$ is obviously the product of the numbers of elements of S_1 and S_2.

Theorem

If S_1, S_2, S_3 are transformation semigroups on X_1, X_2, X_3, then $(S_1 \times S_2) \times S_3 \cong S_1 \times (S_2 \times S_3)$ as transformation semigroup on $X_1 \cup X_2 \cup X_3$.

Proof

Obviously, the map $((s_1, s_2), s_3) \to (s_1, (s_2, s_3))$ is an isomorphism of semigroups $(S_1 \times S_2) \times S_3$ and $S_1 \times (S_2 \times S_3)$. To see that is also so as transformation semigroups, we note as a typical case, $((s_1, s_2), s_3)(x_1) = (s_1, s_2)(x_1) = s_1(x_1)$, and also, $(s_1, (s_2, s_3))(x_1) = s_1(x_1)$.

Definition

The **wreath product** of S_1 with S_2, written $S_1 \varsigma S_2$ is the transformation semigroup on $X_1 \times X_2$ consisting of elements θ on $X_1 \times X_2$ which are given by $\theta : X_1 \times X_2 \to X_1 \times X_2$ such that, $\theta(x_1, x_2) = (s_{1,x_2}(x_1), s_2(x_2))$, with s_2 in S_2 and for each s_{1,x_2} in S_1, s_{1,x_2} s_{1,x_2} being determined by x_2.

It follows from the definition that if S_1, S_2, X_1, X_2 are finite, then $|S_1 \varsigma S_2| = |S_1|^{|X_2|} \times |S_2|$, where $|S_i|$ and $|X_2|$ denote the numbers of elements of S_i (i=1,2) and X_2 respectively.

An alternative description of the wreath product in terms of direct products is given below:

Theorem

If (S_1, X_1) and (S_2, X_2) are transformation semigroups, then $(S_1 \varsigma S_2, X_1 \times X_2) \cong \left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2}\right) \times S_2, \left(\underset{x_2 \in X_2}{\cup} X_{1,x_2}\right) \cup X_2\right)$ where each $S_{1,x_2} \cong S_1$ and $X_{1,x_2} = X_1$

Proof

Define $\phi : (S_1 \varsigma S_2, X_1 \times X_2) \to \left(\underset{x_2 \in X_2}{\times} S_1, x_2 \times S_2, \underset{x \in X_2}{\cup} X_1, x_2\right)$ by

$\phi(\theta_{(\sigma,\tau)}) = (\{\sigma_{x_2}\}, \tau)$ and define

$$\psi\left(\underset{x_2\in X_2}{\times} S_{1},x_2 \times S_2,\ \underset{x\in X_2}{\cup} X_{1},x_2\right) \to (S_1 \varsigma S_2, X_1 \times X_2) \text{ by}$$

$$\psi\ (\{\sigma_{x_2}\},\tau)=\theta_{(\sigma,\tau)}.$$

It follows from the definitions Φ and ψ are inverses of each other, so that both ϕ and ψ are 1-1 and onto.

Now, $(\theta_{(\sigma,\tau)}\ \theta'_{(\sigma',\tau')})\ (x_1, x_2)$

$= \theta_{(\sigma,\tau)}\ (\sigma'_{x_2}(x_1),\ \tau'\ (x_2))$

$= (\sigma_{\tau'(x_2)}\sigma'_{x_2}(x_1)),\ \tau(\tau'\ (x_2)))$

$= (\sigma_{\tau'(x_2)}\sigma'_{x_2})(x_1),\ (\tau\tau')\ (x_2))$

$= (\sigma''_{x_2}(x_1), \tau\tau'\ (x_2))$

$= \theta_{(\sigma'',\tau\tau')}(x_1, x_2)$

where $\sigma''_{x_2}(x_1) = (\sigma_{\tau'(x_1)}\sigma'_{x_2})(x_1)$

Hence

$(\phi(\theta_{(\sigma,\tau)}\theta'_{(\sigma',\tau')}))\ (x_{1,}\ _{x_2})$

$= \phi\ (\theta_{(\sigma'',\tau\tau')})\ (x_{1,}\ _{x_2})$

$= (\{\sigma''_{x_2}\}, \tau\tau')\ (x_{1,}\ _{x_2})$

$= \sigma''_{x_2}\ (x_1), \tau\tau'(x_2)$

$= ((\sigma_{\tau'(x_2)}\ \sigma'_{x_2}(x_1), \tau\tau'(x_2))$ (1)

Also

$(\phi(\theta_{(\sigma,\tau)}),\ \phi\ (\theta'_{(\sigma',\tau')}))(x_{1,}\ _{x_2})$

$$= (\phi(\theta_{(\sigma,\tau)}), (\sigma'_{x_2}(x_1), \tau'(x_2))$$

$$= (\sigma_{\tau'(x_2)}(\sigma'_{x_2}(x_1)), \tau(\tau'(x_2)))$$

$$= ((\sigma_{\tau'(x_2)}\sigma'_{x_2}(x_1), \tau\tau'(x_2)) \qquad (2)$$

Also,

$$= (\phi(\theta_{(\sigma,\tau)}\theta'_{(\sigma',\tau')})) (x_2)$$

$$= (\phi(\theta_{\sigma'',\tau\tau'})) (x_2)$$

$$= (\tau\tau')(x_2)$$

And

$$= (\phi(\theta_{(\sigma,\tau)}\theta'_{(\sigma',\tau')})) (x_2)$$

$$= (\phi(\theta_{(\sigma,\tau)}) (\tau'(x_2))$$

$$= \tau(\tau'(x_2))$$

$$= (\tau\tau')(x_2)$$

It follows from (1) and (2) that

$$(\phi(\theta_{(\sigma,\tau)}\theta'_{(\sigma',\tau')}))= (\phi(\theta_{(\sigma,\tau)}), \phi(\theta'_{(\sigma',\tau')})).$$

Thus ϕ is a homomorphism.

Therefore ϕ is an isomorphism and

$$(S_1 \varsigma S_2, X_1 \times X_2) \cong \left(\left(\underset{x_2 \in X_2}{\times} S_1, x_2 \times S_2, \underset{x \in X_2}{\cup} X_1, x_2\right) \cup X_2\right).$$

Clearly, this description also holds for wreath products of transformation groups.

Wreath products are very much useful when one has a situation of transformation groups or (resp. transformation semigroups) on a union of collection of exactly similar classes, or sets with structures. Applications of wreath products of groups and semigroups appear in Majumdar and Sultana [55] in case of lattices and Majumdar [50], Hossain [52] and Majumdar, Hossain and Dey [53] in case of directed graphs specially, directed rooted trees.

We explain and illustrate applications of direct products and wreath products through the following example.

Example

Let X {a, b, c, d e, f } The full transformation semigroup E(X) consists of all mappings of X into itself, i.e., under $\varphi \in E(X)$, each of a, b, c, d, e, f may be mapped onto any one of a, b, c, d, e ,f ,of course, φ determines which one will be mapped onto which element. Thus the transformation semigroup (E(X), X) has 5^5 =46656 such elements , i.e., $|E(X)| = 46656$ with $|X| = 6$.

Next, write $X = X_1 \cup X_2$, where X_1 = {a, b}, X_2 = {c, d, c, f}. Then, if we consider the semigroups of all those endomappings of X which map X_1 into X_1 and X_2 into X_2. Then S is precisely the direct product $E(X_1) \times E(X_2)$ transforming $X_1 \cup X_2$ into itself so each X_i is mapped into itself. Hence the transformation semigroup (S, X) = $(E(X_1) \times E(X_2), X_1 \cup X_2)$ has $2^2 \times 4^4 = 4 \times 256 = 1024$ elements. Thus, $|E(X_1) \times E(X_2)| = 1024$, with $|X| = |X_1 \cup X_2| = 5$

Finally, write $X = E_1 \cup E_2 \cup E_3$, where E_1 = {a,b}, E_2 = {c, d} and E_3 ={c, f}. Consider the set of all endomappings of X which maps all elements of an E_i into an E_j i,j not necessarily distinct. They form a subsemigroup S′ of E (X) is precisely of the form E(2) ς E (3). Here E(n) is the full transformation semi group on a set with n elements. This is explained as follows. We may write $X_1 = \{x_{11}, x_{21}\}$, $X_2 = \{x_{12}, x_{22}\}$, $X_3=\{x_{13}, x_{23}\}$, i.e., we write $a = x_{11}$, $b=x_{21}$, $c =x_{12}$, $d=x_{22}$, $c =X_{13}$ $f = x_{23}$. Again we may write $Y = \{y_1, y_2\}$ and $X = Y\times \{1,2,3\}$, so that $(y_{i,j}) = x_{i,j}$,i=1,2, j=1, 2, 3,. Then S′ is precisely E (Y) ς E ({1, 2, 3}). Because, under the action of the transformations in S′, each of the $Y\times\{1\}$, $Y\times\{2\}$ and $Y\times\{3\}$ is mapped arbitrarily into any of these yielding the semigroup E ({1, 2, 3}), and each such map is to be considered together with independent maps of each of the sets $Y\times\{j\}$, j = 1, 2, 3 into itself, i.e., with a copy of E(Y) , the total number of required endomappings in S′ is $|E(Y)|^3 \times |E(\{1, 2, 3)\}| = (2^2)^3 \times 3^3 = 4^3 \times 3^3 = 64\times 27 = 1728$.

Remarks

(i) If (S_1, X_1) and (S_2, X_2) are transformation semigroups with $S_2\{1_{x_2}\}$, then both $(S_1 \times S_2, X_1 \cup X_2)$ and $(S_1 ς S_2, X_1 \times X_2)$ may be identified with (S_1, X_1) and $\prod_{x_2\in X_2} S_{1,x_2}, X_1$ respectively, ignoring the trivial action of S_2 on X_2. Thus, in this case, $S_1 \times S_2 \cong S_1$ and $S_1 ς S_2 \cong \left(\prod_{x_2\in X} S_{1,x_2}\right)$ (direct product) as semigroups.

(ii) If $S_1\{1_{x_1}\}$, then both $(S_1 \times S_2, X_1 \cup X_2)$ and $(S_1 ς S_2, X_1 \times X_2)$ may be identified with (S_2, X_2) since $s_{1,x_2}=s_{1,x_2'} = 1_{x_2}$, for each pair of elements $x_2, x'_2 \in X_2$.

(iii) If S and S′ are transformation semigroups on the same set X, then $(S_1 ς S', X \times X)$ may be identified with $\left(\left(\prod_{x\in X} S_x\right)\times S', \bigcup_{x\in X} X_x \cup X\right)$. As semigroups, $S ς S' \cong \left(\prod_{x\in X} S_x\right)\times S'$.

If, in particular, S=S′ and X is finite with $|X|$ =n, then $S\varsigma S \cong . \underline{S \times S \times S \times \cdots \times S \times S}$ (n + 1 copies).

We shall now establish a result which will show that wreath product is distributive over direct product.

Theorem

Let (S_1, X_1), (ς_2, X_2) and (S_3, X_3) be three transformation semigroups. Then $(S_1\varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \cong ((S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$

Proof

Define $\varphi : (S_1\varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \to ((S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$ by $\varphi(\theta) = (\theta', \theta'')$ (1)

where, if $\theta(x_1, x_2) = (\sigma_{1,x_2}(x_1), (\sigma_2, \sigma_3)(x_2))$

$$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$$

$$\theta(x_1, x_3) = (\sigma_{1,x_3}(x_1), (\sigma_2, \sigma_3)(x_3))$$

$$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3)) \qquad (2)$$

Then $(\theta', \theta'')(x_1, x_2) = \theta'(x_1, x_2)$

$$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2)) \qquad (3)$$

$$(\theta', \theta'')(x_1, x_3) = \theta''(x_1, x_3)$$

$$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3)) \qquad (4)$$

Also define

$\psi: (S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3)) \to$

$(S_1\varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3))$

by $\psi(\theta,\theta')=\theta''$

where if $(\theta, \theta')(x_1, x_2) = \theta'(x_1, x_2)$

$=(\sigma_{1,x_2}(x_1), \sigma_2(x_2))$

$(\theta, \theta')(x_1, x_3) = \theta'(x_1, x_3)$

$=(\sigma_{1,x_3}(x_1), \sigma_3(x_3))$ (8)

then

$\theta''(x_1, x_2) = (\sigma_{1,x_2}(x_1), (\sigma_2, \sigma_3)(x_2)$

$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$ (9)

$\theta''(x_1, x_3) = (\sigma_{1,x_3}(x_1), (\sigma_2, \sigma_3)(x_3)$

$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3))$ (10)

It follows from (1) --- (10) that

$\varphi\psi = 1_{((S_1\varsigma S_2)\times(S_1\varsigma S_3),(X_1\times X_2)\cup(X_1\times X_3))}$

and $\psi\varphi = 1_{(S_1\varsigma(S_2\times S_3),X_1\times(X_2\cup X_3))}$

Thus both φ and ψ are 1-1 and onto.

Now, Let θ, $\bar{\theta} \in (S_1\varsigma(S_2 \times S_3), X_1 \times (X_2 \cup X_3))$ by given by

$\theta(x_1, x_2) = \sigma_{1,x_2}(x_1), \sigma_2(x_2))$

$\bar{\theta}(x_1, x_3) = \sigma_{1,x_3}(x_1), \sigma_3(x_3))$

where $\sigma_2 \in S_2$, $\sigma_3 \in S_3$, and σ_{1,x_2}, $\sigma_{1,x_3} \in S_1$

the former being determined by x_2 and the latter by x_3.

and

$$\bar{\theta}\,(x_1, x_2) = \overline{\sigma_{1,x_2}}\,(x_1), \sigma_2\,(x_2))$$

$$\bar{\theta}\,(x_1, x_2) = \overline{\sigma_{1,x_3}}\,(x_1), \sigma_2\,(x_3))$$

where $\bar{\sigma}_2 \in S_2$, $\bar{\sigma}_3 \in S_3$, and $\overline{\sigma_{1,x_2}}$, $\overline{\sigma_{1,x_3}} \in S_1$, the former being determined by x_2 and the latter by x_3.

Then

$$(\theta\ \bar{\theta})\,(x_1, x_2) = (\sigma_{1,x_2}\overline{\sigma_{1,x_2}}(x_1), \sigma_2\bar{\sigma}_2(x_2)),$$

$$(\theta\ \bar{\theta})\,(x_1, x_3) = (\sigma_{1,x_1}\bar{\sigma}_{1}, x_3^{(x_1)}, \sigma_3\bar{\sigma}_3(x_3))$$

Since

$$\varphi\,(\theta) = (\theta_1', \theta_2') \text{and} \varphi(\bar{\theta}) = (\bar{\theta}'_1, \bar{\theta}'_2),$$

Where

$$(\theta_1',\ \theta_2')\ (x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2)),$$

$$(\theta_1', \theta_2')\ (x_1, x_3) = (\sigma_{1,x_2}(x_1), \sigma_3(x_3)),$$

and

$$(\bar{\theta}'_1,\ \bar{\theta}'_2)\ (x_1, x_2) = (\bar{\sigma}_{1,x_2}(x_1), \bar{\sigma}_2(x_2))$$

$$(\bar{\theta}'_1,\ \bar{\theta}'_2)\ (x_1, x_3) = (\bar{\sigma}_{1,x_3}(x_1), \bar{\sigma}_3(x_3))$$

we have $\varphi\,(\theta\ \bar{\theta}) = \varphi\,(\theta)\,\varphi(\bar{\theta})$,

Hence φ is a homomorphism. Therefore φ is an isomorphism. Thus

$(S_1 \varsigma\ (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \cong ((S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3)$.

Chapter 3

Maximal and Minimal Sums and Products

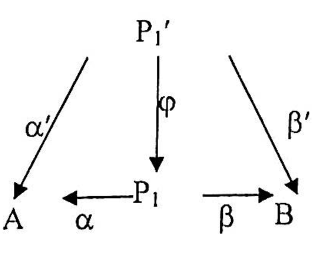
Introduction

We shall construct from two objects in a category a third object which will be maximal or minimal in some sense. The construction will be done in a number of ways and the resulting object will be termed a kind of sum or product of the given two objects. We shall examine the existence and uniqueness of such products, and in possible cases, give examples in certain particular categories. A brief study of their properties will be made in a few cases. For groups two other kinds of constructions will also be considered.

1. Let C be a category.

(a) Definition

(a) Let A and B be two objects in C. An object P_1 will be called a product of the first kind for A and B if (i) there exist epics $P_1 \xrightarrow{\alpha} A$, $P_1 \xrightarrow{\beta} B$ (ii) if P_1' is any object in C such that there epics $P_1' \xrightarrow{\alpha'} A$, $P_1' \xrightarrow{\beta'} B$, then there is an epic $\varphi: P_1' \to P_1$ such that $\alpha\varphi=\alpha'$, $\beta\varphi=\beta'$, i.e. the diagram



is commutative.

In general, let $\{A_i\}$ be a class of objects in C. An object P in C will be called a **product of the first kind** for $\{A_i\}$, if (ii) for each i, there exist an epic $\alpha_i:P_I \to A_i$ and (ii) if there is an object P and for each i, there is an epic $\alpha_i' : P_I' \to P_i$. then there exists an epic $\varphi : P_I' \to P_I$ such that $\alpha_i' = \alpha_i\varphi$ for each i.

If φ is unique, the product P_I, if it exists, is unique upto equivalence.

Examples

(i) If C=S, then the category of sets, $P_I = \underset{\alpha}{\times} A_\alpha$, the cartesian product.

(ii) If C=G, the category of groups, then $P_I = \prod_i A_i$, the restricted direct product of A_i's. $\{\{a_i\} \mid a_i \in A_i\}$ with component wise multiplication with α_i $(\{a_i\}) = a_j$, i.e., α_i is the projection onto the i-th component.

(iii) If C= M_R. the category of all R-modules, R being a ring with 1, then P_I=the direct product of the A_i's $=\{\{a_i\} \mid a_i \in A_i\}$, with component-wise addition and R-multiplication and α_i is the projection onto the i-th component.

(iv) If V_F is the category of vector spaces over a field F, then $P_I = \{\{v_i\} \mid v_i \in V_i\}$, with component wise addition and scalar multiplication with the projection onto the i-th component.

(v) If C= T= the category of all topological spaces, then $P_I = \prod_i A_i$, the product space of the A_i's and α_i is the projection onto the i-th component.

Let $\{A_i\}$ and $\{B_i\}$ be two class of objects in C with P_1 and P_1' as their products of the first kind on C, and let $f_i : A_i \to B_i$ be an epic, for each i. Then, for each i, $f_i\alpha_i : P \to B_i$ and $f_i\alpha_i' : P' \to B_i$ are epics. Hence by the definition of the product there is an epic $\varphi' : P \to P'$ such that $\varphi' f_i\alpha_i = f_i\alpha_i'$.

(b) Definition

If C is such a category of finite sets/finite groups / finite topological spaces, then the product does not exist for infinite class of objects. If in the above definition we remove the restriction of uniqueness from φ, then the product P_1, so defined will be called a **semiweak product** of the first kind.

Examples

(i) Every product of the first kind is obviously semi-weak product of the first kind.

(c) Definition

An object P_2 will called a weak product of the first kind for a class of objects $\{A_i\}$ if there exist epics $\alpha_i : P_2 \to A_i$, for each i and an epic $\varphi : P_2' \to P_2$.

Examples

(i) Clearly every product and every semi-weak product of the first kind are weak products of the first kind.

Comments

(i) It remains to be decided whether the three concepts of a product, a semi weak product and a weak product are all distinct .We are yet to find an example of a semi-weak product which is not a product and an example of a weak product which is not a semi-weak product. In fact we have not been able to provide an example of a weak product which is not even a product.

Also it remains to be decided whether a product defined here is different from the usual concept of product (an universal object) in categories, the given examples of our product are all products in the usual sense.

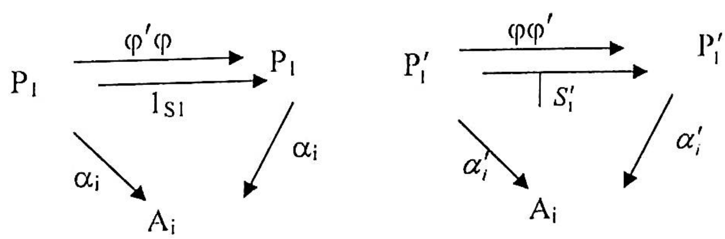
(ii) The products in certain categories like G, M_R, V, T represent objects which are minimal object in same sense.

Theorem

Let $\{A_i\}$ be a class of objects. If a product of the first kind exists for $\{A_i\}$, then it is unique.

Proof

Let P_1 and P_1' be two product of the first kind for $\{A_i\}$. Then for each i, there exist epics $P_1 \xrightarrow{\alpha_i} A_i$ and $P_1' \xrightarrow{\alpha_i'} A_i$ and also unique epics $P_1 \xrightarrow{\varphi} P_1'$ and $P_1' \xrightarrow{\varphi'} P_1$ such that $\alpha_i'\varphi=\alpha_i$ and $\alpha_i\varphi'=\alpha_i'$ for each i. So $\alpha_i\varphi\varphi'=\alpha_i$ and $\alpha_i'\varphi'\varphi=\alpha_i'$, for each i. It follows that the diagram



are commutative. By the definition of a product of the first kind $\varphi'\varphi = 1_{P_1}$ and $\varphi\varphi' = 1_{P_1'}$. Hence both φ and φ' are equivalence.

2. Definition

For a category C, let $\{A_i\}$ be a class of objects and let S_I denote the object in C such that (i) for each i, there exists a monic $A_i \xrightarrow{\alpha_i} S_I$ and if S' is an object in C such that there are monics $A_i \xrightarrow{\alpha'_i} S_I$ and if S' is an object in C such that there are monics $S_I \xrightarrow{\varphi} S'$ such that $\varphi\alpha_i=\alpha'_i$. S_I will be called a **sum of the first kind** for the class $\{A_i\}$.

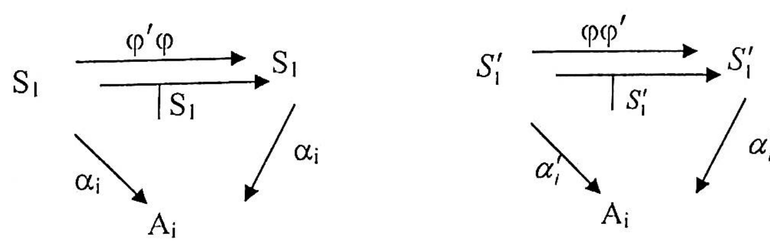
A semi-weak sum and a weak product of $\{A_i\}$ are defined as the corresponding products as S_I with the arrows reversed and 'monic' replacing 'epic'.

Theorem

Let $\{A_i\}$ be a class of objects. If a sum of the first kind exists, then it is unique.

Proof

Let S_I and S'_I be two products of the first kind for $\{A_i\}$. Then for each i, there exist monics $A_i \xrightarrow{\alpha_i} S_I$ and $A_i \xrightarrow{\alpha'_i} S'_I$ also unique monics $S_i \xrightarrow{\varphi} S'_I$ and $S_i \xrightarrow{\varphi'} S'_I$ such that $\alpha'_i\varphi=\alpha_i$ and $\alpha_i\varphi'=\alpha_i'$. Hence $\alpha_i\varphi\varphi'=\alpha_i$ and $\alpha'_i\varphi'\varphi=\alpha'_i$, for each i, thus the diagram



are commutative. By the definition of a sum of the first kind $\varphi'\varphi = 1_{S_I}$ and $\varphi\varphi' = 1_{S'_I}$. Hence both φ and φ' are equivalences.

Examples

(i) For S, $S_I = X_i A_i$, the Cartesian product.

(ii) For G, $S_I = \Sigma Ai$, the restricted direct product of the A_i's, i.e. $\{\{a_i\} \mid a_i \in A_i\}$ with A_i not equal to the identity element only for a finite number of i's and with multiplication defined componentwise.

(iii) For both M_R and V_F, S_I is the direct sum of $\{A_i\}$.

(iv) For T, S_I is equal to the product P_I which is the product space $\prod_i A_i$.

3. Let C be a category. Let $\{A_i\}$ be a class of objects in C. An object P_2 of C will be called a **product of the second kind** for $\{A_i\}$ if (i) there exist monics $P_2 \xrightarrow{\alpha_i} A_i$, and (ii) if there is an object P_2' and there are monics $P_2' \xrightarrow{\alpha_i'} A_i$ then there exists a monic $P_2' \xrightarrow{\varphi} P_2$ such that $\alpha_i \varphi = \alpha_i'$ for each i.

Remark

If we replace 'monic' by 'morphism' and demand φ to be unique then , 'the product of the second kind' becomes 'the product' in the usual sense.

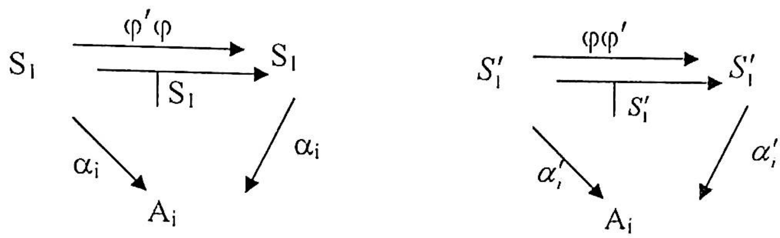
Theorem

It $\{A_i\}$ is a class of objects of C for which a product of the second kind exists, then this product is unique upto equivalence.

Proof

Let P_2 and P_2' be two products of the first kind for $\{A_i\}$. Then for each i, there exist monics $A_i \xrightarrow{\alpha_i} S_I$ and $A_i \xrightarrow{\alpha_i'} S_I'$ also unique monics

$S_I \xrightarrow{\varphi} S'_I$ and $S_I \xrightarrow{\varphi'} S'_I$ such that $\alpha'_i\varphi=\alpha_i$ and $\alpha_i\varphi'=\alpha_i'$. Hence $\alpha_i\varphi\varphi'=\alpha_i$ and $\alpha'_i\varphi'\varphi=\alpha'_i$, for each i, thus the diagram



are commutative. By the definition of P_2 and P_2' it following that $\varphi\varphi'= 1_{P_2}$ and $\varphi'\varphi = 1_{P_2}$ i.e. φ and φ' are equivalences, since α_i and and. Hence both φ and φ' are monics.

Examples

(i) For S, $S_I = \times_i A_i$, the cartesian product.

(ii) For G, $S_I = \sum A_i$, the restricted direct product of the A_i's, i.e. $\{\{a_i\} \mid a_i \in A_i\}$ with A_i not equal to the identity element only for a finite number of i's and with multiplication defined component wise.

(iii) For both M_R and V_F, S_I is the direct sum of $\{A_i\}$.

(iv) For T, S_I is equal to the product P_I which is the product space $\prod_i A_i$.

Comments

(i) The product of the second kind does not always exist. For example Z_2 and Z both are Z-modules or abelian groups. and there is no product of this second kind for them in M_Z=A.

(ii) As known by example (i) above, in certain categories the product of the second kind is a maximal object (a set) which is contained in each of the given objects.

(iii) The products of the two kinds introduced here are thus dual to each other in some sense.

4. We now consider a second kind of sum in a category. Let C be a category and $\{A_i\}$ a class of objects in C. An object S_2 will be call a product of the second kind for $\{A_i\}$ if (i) there exists an epic $A_i \xrightarrow{\alpha_i} S_2$ for each i, and (ii) if S_2' is an object such that there is an epic $A_i \xrightarrow{\alpha_i'} S_2'$, then there is an epic $S_i \xrightarrow{\varphi} S_2'$, such that $\alpha_i\varphi = \alpha_i'$.

Remark

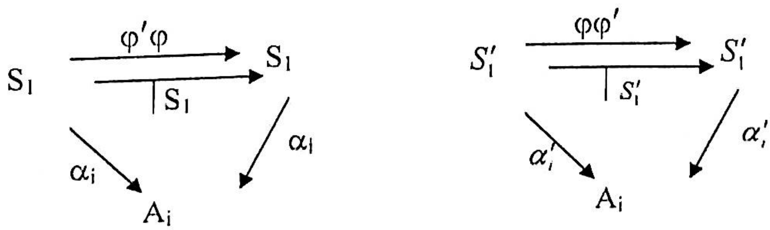
If epic is replaced by morphism and φ is assumed to be unique, the sum of the second kind becomes the usual coproduct.

Theorem

If $\{A_i\}$ be a class of objects in A category. If a sum of the second kind exists for $\{A_i\}$, then it is unique upto equivalence.

Proof

Let S_2 and S_2' be two sums of the second kind for $\{A_i\}$. Then for each i, there exist epics $A_i \xrightarrow{\alpha_i} S_2$ and $A_i \xrightarrow{\alpha_i'} S_2'$,and also there exist epics $A_i \xrightarrow{\varphi} S_2'$ and $A_i \xrightarrow{\varphi'} S_2'$ such that $\varphi\alpha_i' = \alpha_i$ and $\varphi'\alpha_i = \alpha_i'$ for each i. Hence $\varphi'\varphi\alpha_i = \alpha_i = 1_{S_2}$ and $\varphi\varphi'\alpha_i' = \alpha_i' = 1_{S_2'}$ for each i, i.e., the diagram



are commutative. By the definition of S_2 and S_2' $\varphi'\varphi = 1_{S_2}$ and $\varphi\varphi' = 1_{S_2'}$,

since α_i and α_i' are epics. Hence both φ and φ' are equivalences.

Examples

(i) If $A_1 = C_6$ and $A_2 = S_3$, then the product of the second kind for $\{A_1, A_2)$ is C_2.

(ii) If C_{m_1}, C_{m_2} ... C_{m_r} are cyclic groups of order m_1, m_2, ... m_r with the h.c.f of m_1, m_2, ... m_r = p a prime, then C_p is the sum of the second kind for C_{m_1}, C_{m_2} ... C_{m_r}.

5. (a) We now consider only G, the category of groups.

Definition

For two groups A and B we call a **sum of the** third kind for A and B if (i) Q contains A and B as subgroups and (ii) no proper subgroup of Q contains both A and B. Q is thus a minimal group containing A,B as subgroup.

The definition is similar for an arbitrary class $\{A_i\}$ replacing A and B.

Comments

(i) It is easily seen that Q need not be unique. For if $A=C_3$ and $B=C_2$, the cyclic groups of order 3 and 2 respectively, then both S_3, the semidirect product of A and B, and C_6, the direct product of C_3 and C_2, are sums of the third kind for A and B.

(ii) In general, for any two groups A and B, their direct product and semidirect product both are sums of the third kind for A and B.

(a) The previous examples also show that the product of the second kind for A and B need not be abelian even though both A and B are abelian.

(iii) If we impose A Q in the definition of the product Q, the totality of such Q's include as particular cases the extensions $1 \to A \to Q \to B \to 1$, and in these cases, all these extensions build up the second cohomology group H^2 (G,Z) with integral coefficients. In such situations the further restriction is $B \cong \frac{Q}{A}$.

If A, B are finite, consideration of the orders of A, B and Q proves that such Q is induced a sum of the third kind for A and B.

(iv) The free product of two groups A and B, two is a product of the second kind for A and B. We thus see that if A and B both are finite, a product of the second kind for A and B need not be finite.

(b) The corresponding sum in the category of sets is the union. $A \cup B$ is the sum of second kind for sets A and B. For an arbitrary class of sets $\{A_i\}$, this sum Q is $\bigcup_i A_i$.

If in G, A and B are subgroups of a group G, then the sum Q of the third kind for A and B is the group theoretic union of A and B. This means that Q is the subgroup of G generated by $A \cup B$. Q for $\{A_i\}$ where are subgroups of a group, is the subgroup generated by $\bigcup_i A_i$.

(c) In the category M_R and V_F, the sum of the third kind for A and B, is their direct product $A \oplus B$. If A, B are submodules of a module or subspace of a vector space V, then their sum Q is A+B in M_R or V_F respectively.

(d) In the category T, sum Q for two disjoint topological spaces A and B is their direct sum and if A and B are two intersecting topological spaces for which the intersection has the same topology as subspaces of A and B, then the sum Q is the sum A+B defined in **Chapter 2**. We have discussed its properties there.

(e) In the category of lattices a product of the third kind for a class of lattices $\{A_i\}$is a lattice S_3 such that (i) each A_i is a sublattice of S_3, and (ii) S'_3 is a lattice containing each A_i as a sublattice, then S_3 is a sublattice of S'_3.

Comment

(i) A sum of the third kind for a class of lattices need not always be unique.

For if A_1 and A_2 both are chains of length 3, then A_1, A_2 themselves as well as the lattices are sums of the third kind for A_1 and A_2 and these are non isomorphic lattices.

(ii) If the A_i's are sublattices of a given lattice L, then S_3 for $\{A_i\}$ is the sublattice of L generated by $\bigcup_i A_i$.

6. (a) Again we consider only the category of groups. Let A and B be two groups. Let R denotes a group which has both A and B as factor groups but no proper factor group of R has this property. We call R a **product of the third kind** for A and B.

The definition for a product of the third kind for an arbitrary class $\{A_i\}$ of groups is similar.

Examples

(i) If $A=C_4$, $B=C_6$, cyclic groups of order 4 and 6 respectively, then C_{12} and $C_4 \times C_6$ both are products of the third kind for $\{A, B\}$. Thus a product of the third kind may not always be unique.

(ii) Let $A_i = C_{p_i}$, i =1,2, ... then a product of the third kind for $\{A_i\}$ is the quasi-cyclic group (Z_p), the group with generators, $x_1, x_2, x_3 \ldots = \{x_n \mid n \in N\}$ and defining relations. $x_1^p = 1,\ x_1 = x_2^p,\ x_2 = x_3^p, \ldots, x_n = x_{n+1}^p = \{x_1^p = 1,\ x_n = x_{n+1}^p \mid n \in N$

(b) In V_F, the corresponding product of V_1 and V_2 is V_1, if V_1 and V_2 are finite dimensional with $\dim V_1 \geq \dim V_2$.

(c) (i) We now consider a product of the third kind in T, defined as in (a). If X and Y are topological space none of which is homeomorphic to a quotient space of the other, then $X \times Y$ is a product of the third kind for X and Y.

(ii) If X is a topological space, then a product of the third kind for X^m and X^n is X^m, if $m \geq n$. In particular, for $A=R^m$, $B=R^n$ ($m \geq n$), A is a product of the third kind for A and B.

Chapter 4

Semi-periodic Product

Introduction

Two kinds of products of partially ordered sets have been defined. These have been called semiperiodic product and periodic product. These products for lattices have been studied and a few properties have been established.

Definition

A partially ordered set $(X,\leq)$ will be called a **semi-periodic product** of a partially ordered set $(A, \leq_A)$ with a collection of partially ordered sets $B=\{B_a, \leq_a\}$ $a \in A$ if (i) there exists an onto order homomorphism $p:X\rightarrow A$, such that $p^{-1}(a) \cong B_a$, for each $a\in A$. (ii) for each $a \in A$, there exist order homeomorphisms i_a: $B_a \rightarrow X$ and $\pi_a:X\rightarrow B_a$ such that $\pi_a i_a = 1B_a$ (iii) for each pair $a, a' \in A$, $a \leq a'$ and $a \neq a'$ together implies $x\leq x'$ whenever $x \in p^{-1}(a)$, $x' \in p^{-1}(a')$ and if $a=a'$, then $x \leq x'$ if $f_a(x) \leq f_{a'}(x')$ X is denoted by A □ B. X will be said to be **semi-periodically ordered.**

Periodic Product

If each B_a is order-isomorphic to a partially ordered set $(B, \leq_B)$, then X is said to be periodically ordered. In this case, X is called the **periodic product** of A with B and is denoted by A □ B.

Examples

1. Let $(A, \leq_A)$ be a partially ordered set and $B= \{B_a\}$ where $(B_a, \leq_a)$ is a bounded partially ordered set. Let $X= \{(a, b_a), a\in A, b_a\in B_a\}$ and define $\leq$ on X by $(a, b_a) < (a', b'_{a'})$ if either $a \leq_A a'$ or $a = a'$ and $b_a <_a b'_{a'}$. Then $X = A \square B$. Here $p: X \to A$ is given by $p(a,b_a) = a$, and $i(a) = (a, 0_{B_a})$, for each $a \in A$, $b_a \in B_a$ while $\pi_a : X \to B_a$ and $i_a : B_a \to X$, are given by $\pi_a(a,b_a) = b_a$, $i_a(b_a) =(b_a^{`},a)$ for each $a\in A$, $b_a \in B_a$.

In particular, if each B_a is order-isomorphic to a partially ordered set B, then $A \square B_a = A \square B$ is order isomorphic to the direct product $A \times B$. In particular if $A=[0, 1] = B$, then $A \times B$, the closed unit square in the Euclidean plane is periodically ordered.

Also if $A=S'$, the open unit circle $= \{(1, \varphi, 0)| \ 0 \leq \varphi < 2\pi\}$ in cylindrical polar coordinates, and $B=[0, 1]$, then the $X=A\times B$ is the vertically half-open right cylindrical surface with a circular bare S^1. In cylindrical polar coordinates $X= \{(1, \varphi, z)| \ 0 \leq \varphi < 2\pi, 0 \leq z \leq 1\}$. $(1, \varphi, z) \leq (1, \varphi', z')$, if and only if $z \leq z'$ or $z = z'$, $\varphi \leq \varphi'$. After a regular period of 2π in the variation of φ; each point on the cylinder returns to the same vertical line ,a fact which motivates the name of the term periodic.

2. Let A be a totally ordered set, and for each $a \in A$, let B_a be a bounded partially ordered set and let $B =\{B_a\}_{a \in A}$. Let A be such that each $a\in A$ has an immediate successor a', i.e., there is an element $a' \in A$ such that $a \neq a'$, $a \leq a'$ and if $a \leq \bar{a}$, for some $\bar{a} \in A$, then either $a = \bar{a}$ or $\bar{a} = a'$. Let $X = (\bigcup_{a\in A} B_a)$,with $0_{B_{a'}} = 1_{B_a} = a$, for each $a \in A$ .Then $X=A\square$ **B**

,where p: X → A, i: A →X, π_a : X→ B_a and i_a : B_a → X are given by p(b_a) = a, i(a) = 1_{B_a} ,π(b_a)= b_a ,i(a) = b_a.

As a particular case, we have the following semiperiodic product A □ B as shown in the figure

Here B ={ B_{a_1} , B_{a_1} , B_{a_1} , B_{a_1} , B_{a_1} }, A={a_1 , a_2 , a_3 , a_4 ,a_5 }.This example is easily seen to be a lattice obtained from a chain product of B_{a_1} , B_{a_1} , B_{a_1} , B_{a_1} , B_{a_1} by identifying 0_{A_i} with $1_{A_{i+1}}$ i=1,2,..4.

3. Let X be the right circular cone with height h, semi-vertical angle α and base the circle x^2+y^2 = 1, z=0 and axis the z-axis. Then, in Cartesian coordinates,

$$X = \left\{ \left(\frac{h-u}{h} \cos v, \ \frac{h-u}{h} \sin v, \ u \right) \middle| 0 \le u \le h, 0 \le v \le 2\pi \right\}.$$

For two points P.Q with the parametric values (u_1, v_1), (u_2, v_2), P ≤ Q if u_1 ≤ u_2 or if u_1=v_2 and v_1 ≤ v_2. Then X=A □ B. Where A is the segment of the line $\frac{x}{1} = \frac{z}{h}$, y=0, between the points (1, 0, 0) and (0, 0, h) and B is the circle x^2+y^2=1, z=0, i.e., B={(cos v, sin v, 0) | 0 ≤ v < 2π} with the natural ordering i:A→X p:X→A are given by i (x, y, z) = (x, y, z), $p\left(\frac{h-u}{h} \cos v, \ \frac{h-u}{h} \sin v, \ u \right)$ = (cos v, sin v, 0).

3. Let X be the segment of a circular helix X={(cos t, sin t, t) | 0 ≤ t < n, 0 ≤ t ≤ 2nπ} winding n times round the right circular cylinder with base x^2+y^2=1, z=0 and axis the z-axis. Then X = A □ B, where A={0, 1, 2, ... n-1} (with one point removed) i.e., B={(cos θ, sin θ, 0) | 0 ≤ θ < 2π} with ordering on B induced by that on θ and the order of A is natural.

Here p (cos t, sin t, t) = k, if t $\in$ [2kπ, (2k+1) π) and for $0 \leq k \leq n-1$ and $0 \leq t < 2\pi$, π_k (cos t, sin t, t) = (cos t, sin t, 0) i_k (cos t, sin t, t) = (cos t, sin t, (2k-1) π+t). Hence $p^{-1}(k)$ = {(cos t, sin t, t) | t $\in$ [2kπ, (2k+1) π)} $\cong$ {(cos t, sin t, 0) | t $\in$ [0, 2π)} (order isomorphic) = B. Also π_k i_k (cos t, sin t, 0) = π_k (cos t, sin t, (2k-1) π+t) =(cos t, sin t, 0). $\therefore \pi_k i_k = 1_B$.

4. Let B={a, b, c, d, e, f, g} be the set of seven musical nodes, viz., do, re, mi, pha, sol, la, ti in an octave and A={1, 2, 3...} is the set of numbers representing the audiable octaves in the ascending order of the pitch. Then the frequencies of the notes in all the octaves has a natural ascending order. Also B has a natural ascending order in accordance with their pitches. If we write the note x in the n-th octave as (x, n) then all the audiable X={(x, n) | x $\in$ B, n $\in$ A}. Let $\overline{(x,n)}$ denote the frequency of the note (x , n). If we write (x, n) $\leq$ (x′, n′) if $\overline{(x,n)} < \overline{(x',n')}$; i.e., if either n $\leq$ n′ or n = n′ and n $\leq$ n′, then X = A □ B.

Here p(x, n)=n, while π (x, n) =x. We may take i(x) = (x, n_0) for a fixed $n_0 \in$ A. In particular, we may write i(x) = (x, 1). Then p^{-1} (n) = {(a, n), (b, n) ... (g, n)} $\cong$ {(a, b, c ... g} = B. Also i(x) = x, so that $\pi i = 1_B$.

In this example, the periodicity of the product is reflected in the repeated return of the same tunes in the same order as one passes from one octave to another. This is an audio property which does not have a counter part in the visual case, where as the frequency of the light wave goes on increasing or decreasing, the corresponding nature of the colour goes on changing in violet → red direction or red-violet direction without any kind of repetition occurring.

Properties of Periodic Product

We shall now prove a few results concerning the properties of a semiperiodic and periodic products.

Theorem 1

If $(A \leq_A)$ and each $(B_a, \leq_a)$ are totally ordered, $A \square B$ is totally ordered.

Proof

Let $x, x' \in X$. Let $p(x) = a$ and $p(x') = a'$ First suppose $a \neq a'$. Since A is totally ordered either $a \leq a'$ or $a' \leq a$. Suppose $a \leq a'$. Then by the definition of a semiperiodic product, $x \leq x'$. Next let $a = a'$.Then $f_{a'}(x)$ and $f_a(x') = f_{a'}(x')$ both belong to B_a. The latter being totally ordered $f_a(x) \leq f_a(x')$ or $f_a(x') \leq f_a(x')$, f_a being an order isomorphism, $x \leq x'$ or $x' \leq x$.

Corollary 1

If A and B are totally ordered sets, so is $A \square B$

Theorem 2

If $(A \leq_A)$ and each $(B_a, \leq_a)$ are complete lattices, then $A \square B$ is a complete lattice.

Proof

Let $x, x' \in X$. Let $u = p(x) \vee p(x')$, $l = p(x) \wedge p(x')$ and $u' = \pi(x) \vee \pi(x')$, $l' = \pi(x) \wedge \pi(x')$. If $p(x) = p(x')$, define $x \vee x' = \inf \pi^{-1}(u')$. $x \wedge x' = \sup \pi^{-1}(l')$.

Let $x, x' \in X$. If $p(x) = p(x') = a$, say then $x, x' \in p^{-1}(a)$. Hence $x \vee x'$ and $x \wedge x'$ are $f_a^{-1}(f_a(x) \vee f_a(x'))$ and $f_a^{-1}(f_a(x) \wedge f_a(x'))$, the latter are well defined elements of X and are indeed equal to $x \vee x'$ and $x \wedge x'$ respectively since B_a is a lattice and f_a is an order-isomorphism. Next let $p(x) \neq p(x')$. Then $x \vee x' = \inf p^{-1}(p(x) \vee p(x'))$ and $x \wedge x' = \sup p^{-1}(p(x) \wedge p(x'))$. The existence of the elements on the right hand side follows from the facts that (i) A is a lattice and that each B_a is a complete lattice.

It is clear from the nature of $x \vee x'$ and $x \wedge x'$ in both the above cases that A □ B is a complete.

Corollary 2

If A and B are complete lattices so is A □ B.

Theorem 3

Let A and B each B_a be bounded complete lattices. Then A □ B is a bounded complete lattice.

Proof

We are only to show that the lattice A □ B is a bounded. Let 1A, 0A 1Ba and 0_{B_a} be the greatest and lowest elements in A and B_a respectively. Then $1_{B_{1A}} = 1_{A \triangleleft B}$ and $0_{B_{0_A}} = 0_{A \triangleleft B}$. The truth of this statement follows from the definition of A □ B.

Corollary 3

If A and B are bounded complete lattices, then so is A □ B.

Theorem 4

If A, B_a are complete lattices from each $a \in A$ and if each B_a is order isomorphic to B. Regard A, B and A □ B as topological spaces with order topology. Then A □ B is a covering space of B.

Proof:

By Theorem 2, A □ B is a complete lattice. We recall that the order topology on a partially ordered set X s the topology generated by all intervals (a, b) on X. Here a, b $\in$ X and (a, b) ={ x$\in$X | $a \leq x \leq b$, $x \neq a$, $x \neq b$ }.

We first prove that each of A, B and A □ B is path-connected. Let a_1, a_2 $\in$A Define f: I$\rightarrow$A by f(x) =a , if $0 \leq x < \frac{1}{2}$

=a $\vee$ b , if $x = \frac{1}{2}$

= b , if $\frac{1}{2} < x \leq 1$

Then f is continuous, to see this U be an open interval in A, If a, b, a$\vee$b $\notin$U, then $f^{-1}(U) = \Phi$, which is open. If a $\in$U, b, a $\vee$ b $\notin$ U, then $f^{-1}(U)$ = [o, x) , for some $x < \frac{1}{2}$. If b $\in$U ,a, a$\vee$b $\notin$U. then $f^{-1}(U)$ = (x, 1], for some $x > \frac{1}{2}$. If a, a $\vee$ b $\in$U, b $\notin$U then $f^{-1}(U)$ = [0,x], for some x, such that $\frac{1}{2} < x \quad < 1$. If

If a$\notin$U, b, a $\vee$ b $\in$U, then $f^{-1}(U)$ = (x, 1], for some x such that $0 < x < \frac{1}{2}$.If a, b, a $\vee$ b $\in$U, then $f^{-1}(U)$ = [0, 1].Thus ,in each case , $f^{-1}(U)$ is open . Hence f is continuous . Also f(0) =1,f(1) =b, So f is indeed a path-connected.

Next, for each $a \in A$, Let f_a:$B_a \to B$ be an order isomorphism. Define g: A□B→B by g(x)= $f_a \pi_a(x)$, for some a, arbitrary but fixed. Let $b \in B$and let U_b be an open internal in B such that $b \in U_b$. Then $f^{-1}{}_a(U_b)$ is an open set V_{ab} in B_a which is oder-isomorphism to U_b. Then $\pi^{-1}{}_a f^{-1}{}_a(U) = \pi^{-1}{}_a(V_{ab})$ is an open set in A □ B. It is clear that the arc-components of $\pi^{-1}{}_a(V_{ab})$ are open intervals in A and each of these is oder isomorphic to V_{ab}, and hence ,to U. Therefore , A □ B is a covering space of B.

Chapter 5

On Connected Sums and External Sums of Topological Spaces

Introduction

Connected sums of surfaces were introduced and applied for classification of compact surfaces by Möbius. Here a similar concept has been introduced for more general situations like metric spaces and topological spaces. Investigation has been made regarding the conservation of various properties of a topological space under connected sum .

1. Given two topological spaces X and Y , if is sometimes possible to define a topology on $X \cup Y$. Such a space $X \cup Y$ may be called a sum of X and Y. It is obvious that the existence of a sum requires the intersection $X \cap Y$ to be either or to have the same topology as subspaces of X and Y . This may be called the compatibility condition.

Majumdar and Asaduzzaman [50] have considered two spaces X and Y satisfying the above condition and called $X \cup Y$ the **sum** of X and Y when the topology on $X \cup Y$ is precisely $\{G \cup H \mid G, H$ open in X and Y respectively . They denoted it by $X + Y$. When $X \cap Y = \phi$, they called the sum direct sum and denoted it by $X \oplus Y$. In the case the expression of an open set in $X \oplus Y$ as a union of open sets in X and Y is unique. They studied some properties of X+Y.

A concept some what similar to a sum of two spaces occur in Bourbaki [31] and the idea of the direct sum occurs in a different name in Dugunji [61].

When X and Y are subspace of a topological space Z, the natural way of considering X as a topological space such that X and Y are its subspaces would be to regard $X \cup Y$ as a subspace of Z i.e., to consider the subspace topology on $X \cup Y$ with this topology on $X \cup Y$ we call $X \cup Y$ the usual or the normal sum of X and Y

Here we shall make use of the concepts of sum, direct sum and normal sum to define new types of amalgamations of topological spaces. These will be called the **connected sum** and the **external sum.**

2. We shall make the union of two metric spaces into a metric space by defining a suitable metric on the union. This is done in the following theorem.

Theorem (A)

Let (X, d_1) and (Y, d_2) be two metric space such that $X \cap Y \neq \phi$ and let

$$d_1 \mid (X \cap Y) \times (X \cap Y) = d_2 \mid (X \cap Y) \times (X \cap Y)$$

Let us define

$$d : (X \cup Y) \times (X \cup Y) \rightarrow \Re$$

be given by $d(z,z') = d_1 (z, z')$ if $z, z' \in X-Y$

$= d_2 (z, z')$ if $z, z' \in X-X$

$= \inf \{d_1(z,c) + d_2(c, z') \mid c \in X \cap Y\}$, if $z \in X-Y$, $z' \in Y-X$

Then d is a well-defined metric on $X \cup Y$. Since

$d_1(z, z') = d_2(z, z')$ for all $(z, z') \in (X \cap Y) \times (X \cap Y)$.

(i) Now $d \geq 0$ since $d_1 \geq 0$ and $d_2 \geq 0$.

(ii) From the definition of d, it is clear that $d(t, t) = 0$ for all $t \in X \cup Y$

(iii) For any t_1, t_2 in $X \cup Y$,

$$d(t_1, t_2) = d_1(t_1, t_2) = d_1(t_2, t_1), \text{ if } t_1, t_2 \in X - Y,$$

$$= d_2(t_1, t_2) = d_2(t_2, t_1), \text{ if } t_1, t_2 \in Y - X,$$

$$= \inf \{d_1(t_1, c) + d_2(c, t_2) \mid c \in X \cap Y\}, \text{if } t_1 \in X - Y, t_2 \in Y - X,$$

$= d(t_2, t_1)$ when any two of the t's are in X one is in Y.

Let $t_1, t_2 \in X$ and $t_3 \in Y$ then $d(t_1, t_2) + d(t_2, t_3) = d_1(t_1, t_2) + \inf \{d_1(t_2, c) + d_2(c, t_3), c \in X \cap Y\} = \inf \{d_1(t_1, t_2) + d_1(t_2, c) + d_2(c, t_3), c \in X \cap Y\} \geq \inf_{c \in X \cap Y} \{d_1(t_1, c) + d_2(c, t_3)\} = d(t_1, t_3)$.

i.e. $d(t_1, t_2) + d(t_2, t_3) \geq d(t_1, t_3)$.

Therefore d is a metric on $X \cup Y$.

3. Connected Sum of Metric Spaces

Definition

Let (X, d), (Y, d') be two disjoint metric spaces such that the closed sphere

$S = S_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$ and $S' = S_{r'}(y_0) = \{y \in Y \mid d'(y, y_0) \leq r'\}$ are homeomorphic. Let $f : S \to S'$ be a homomorphism. If C and C' denote the boundary of S and S′ respectively, then $C=\{x\in X \mid d(x, x_0) = r\}$ and $C'=\{y\in Y \mid d'(y, y_0) = r'\}$. Also suppose that f restricted to C is not only a homomorphism but also an isometry [and so $d(x_1, x_2) = d(f(x_1), f(x_2))$ for every pair of points x_1, x_2 on C] of C into C′. Let Z= (X–int (S)) ∪ (Y–int (S′)). Define a relation ~ on Z as follows :

(i) for each $z \in Z-(C\cup C')$, $z\sim z$

(i) for each $z \in C$, $z\sim z$ and $z\sim f(z)$

(i) for each $z' \in C'$, $z'\sim z'$ and $z'\sim f^{1}(z')$.

Then ~ is an equivalence relation on Z.

Under this identification topology $\frac{Z}{\sim}$ is termed as the **connected sum** of X and Y. Then $\bar{Z} = \frac{Z}{\sim}$ is thus a topological space .We can regard $\bar{Z}$ as $(X\cup Y\text{-}(\text{int}(S) \cap \text{int}(S'))$ under the identification x with f(x) for all $x\in C$. So here we regard $C = C'=(X\text{-int}(S)) \cap (Y\text{-int}(S'))$. We shall investigate whether $\bar{Z}$ inherits a metric from the metric d and d′ or not.

The connected sum of X and Y is denoted by $X\#_F Y$.

Let us define a function $\bar{d}$ on $\bar{Z}$

$$\bar{d}(z_1,z_2) = d(z_1, z_2) \text{ if } z_1, z_2\in(X\text{-int } S) - (Y - \text{int } S)$$

$$= d'(z_1, z_2) \text{ if } z_1, z_2\in (Y-\text{int } S) - (X\text{-int } S)$$

$$= \inf\{d(z_1,c) + d'(c, z_2) \mid c \in C\}, \text{if } z_1\in(X\text{-int } S) - (Y - \text{int } S), z_2\in(Y\text{-int } S) - (X- \text{int } S).$$

Then d is well-defined , since $d|_{C\times C} = d'|_{C\times C}$. Now by Theorem (A) d is a metric on $\overline{Z}$ .Thus $\overline{Z}$ is also a metric space ($\overline{Z}$, d) and d induces the required identification topology on $\overline{Z}$.

Theorem (B)

Let (X, d_1) and (Y, d_2) be compact, then ($\overline{Z}$, d) = X#Y , the connected sum of X and Y is compact.

Proof

To prove that ($\overline{Z}$, d) is compact, it suffices to show that ($\overline{Z}$, d) is sequentially compact. Let $\{z_n\}$ be a sequence in $\overline{Z}$. Then there exist subsequences $\{Z_{n_i}\}$ and $\{Z_{n_j}\}$ of $\{z_n\}$ such that $\{Z_{n_i}\}$ and $\{Z_{n_j}\}$ are sequences in X and Y respectively. Here at least one of these subsequences, say $\{Z_{n_i}\}$, must contain as infinite number of terms of the sequence $\{z_n\}$. Since X is compact, $\{Z_{n_i}\}$ has a convergent subsequence of $\{Z_{n_{i_\alpha}}\}$ which is also a convergent subsequence of $\{z_n\}$. Thus $\overline{Z}$ is compact.

4. Connected Sum of Topological Spaces

Definition

Let (X, $\mathfrak{I}$), (Y, $\mathfrak{I}'$) be two topological spaces such that $X \cap Y = \phi$. Suppose that there exists nonempty closed sets F and F′ of X and Y respectively such that F is homeomorphic to F′. Let f : F→F′ be a homomorphism. Let $\overline{f} = f|b$ (F). Then $\overline{f}$ is a homeomorphism. Let $\overline{f}$: b(F) (=B) →b (F′) = B′. Let Z=(X–int (F)) ∪ (Y–int (F′)). We define a relation ~ on Z as follows :

(i) for each $z \in Z - (B \cup B')$, z~z

(ii) for each $z \in B$, $z \sim z$ and $z \sim f(z)$

(iii) for each $z' \in B'$, $z' \sim z'$ and $z' \sim f^{1}(z')$.

Then ~ is an equivalence relation on Z.

We write $\overline{Z} = \frac{Z}{\sim}$ Then $\overline{Z} = X_1 \cup B \cup Y_1$ where $X_1 = X - F, Y = Y_1 - F'$, and B has been identified with B′ via the homeomorphism $\overline{f}$. We regard $\overline{Z} = X_1 \oplus B \oplus Y_1$, where X_1, B and Y_1 have subspace topologies from X and Y. $\overline{Z}$ will be called **connected sum** of X and Y and will be denoted by $X \#_F Y$.

If X are Y are disjoint subspaces of a topological Z and the topology on the union (X-int S) ∪ (Y-int S′) is the subspace topology, then the resulting connected sum will be called the **usual** or the **normal** connected sum. As is well known for the both compact orientable and non-orientable subspaces ,

(i) The connected sum of two spheres is homeomorphic to a sphere.

(ii) The connected sum of two tori is a sphere with two handles. In general the connected sum of n tori is a sphere with n handles.

(iii) Any orientable compact surface is either homeomorphic to a sphere or to a connected sum of tori onto a connected sum of projective planes. If S_1 and S_2 are projective planes then $S_1 \# S_2$ is a Klein Bottle.

Normal connected sum is useful for describing structures of compact surfaces. Also it is of great help for determination of fundamental groups of surfaces .It is expected that general connected sum will also be of some use in these regards.

Theorem 1

The connected sum $X\#_F Y$ is connected if and only if both X-int (F) and Y- int (F′) are connected.

Before proving the theorem, we recall the definition or sum of two topological spaces. Let X and Y be topological spaces such that either $X \cap Y = \Phi$ or the topology on induced by X and Y are the same. Then $X \cup Y$ is a topological spaces where the topology is $\{G \cup H \mid G$ is open in X and H is open in Y$\}$. Here $X \cup Y$ is called the **sum** of X and Y and denoted by X+Y. We first prove a result due to Majumdar and Asaduzzaman [50].

Lemma 1

X+Y is connected if and only if both X and Y are connected and $X \cap Y \neq \Phi$.

Proof

First, let $X \cup Y \neq \Phi$. Suppose $(X \cup Y.T)$ is not connected. Then there exist two nonempty T-open sets G_1 and G_2 such that $G_1 \cap G_2 = \Phi$ and $X \cup Y = G_1 \cup G_2$. Now $X \cap G_1$ and $X \cap G_2$ are T_1-open sets and $(X \cap G_1) \cap (X \cap G_2) = X \cap (G_1 \cap G_2) = \Phi$ and $(X \cap G_1)$ '$\cup$' $(X \cap G_2) = X \cap (G_1 \cup G_2) = X \cap (X \cup Y) = X$. Since $(X. T_1)$ is connected, one of the sets $X \cap G_1$ and $X \cap G_2$ must be empty. Let $X \cap G_1 = \Phi$ then $X \cap G_2 = X \Rightarrow X \subseteq G_2$. Similarly, if $X \cap G_2 = \Phi$ then $X \cap G_1 = X \Rightarrow X \subseteq G_1$. Thus $X \subseteq G_1$ or $X \subseteq G_2$. Similarly the connectedness of (Y, T_2) $Y \subseteq G_1$ or $Y \subseteq G_2$ then $X \cap Y \subseteq G_1 \cap G_2 = \Phi$. Similarly, $Y \subseteq G_1$ and $Y \subseteq G_2$ implies $X \cap Y = \Phi$. Hence both X and Y are in

G_1 or both are in G_2. If X. $Y \subseteq G_1$ then $G_1 \cup G_2 = X \cup Y \subseteq G_1 \Rightarrow G_2 = \Phi$. Similarly X, $Y \subseteq G_2$ then $G_1 = \Phi$. The contradictions in both the cases prove that $(X \cup Y. T)$ is connected.

Conversely, if $X \cap Y = \Phi$ then $X \cup Y$ is obviously disconnected since X and Y are open in $X \cup Y$.

Proof of Theorem 1

We know that $X\#_F Y = (X - int(F)) + (Y - int(F))$ where F and F′ have been identified via a homeomorphism of F onto F′. Since $=(X - int(F)) \cap (Y - int(F)) = b(F) \neq \Phi$, lemma 1 shows that $X\#_F Y$ is connected if and only if both $=(X - int(F))$ and $(Y - int(F'))$ are connected i.e., if and only if $(X - int(F))$ and $(Y - int(F'))$ are both connected.

We now prove

Lemma 2

If X and Y are locally connected then the sum X+Y is locally connected.

Proof

Let $z \in X+Y$. If W is an open set in X+Y with $z \in W$, then $W = U \cup V$ with U open in X and V open in Y. If $z \in U$, there exist connected open sets U' in X with $z \in U'$ and $U' \subseteq U$. Since U and U' are open in X+Y, the latter is locally connected.

Lemma 3

If X is locally connected and R is an equivalence relation on X, then the quotient space $\frac{X}{R}$ is locally connected.

Proof

Let π: X $\rightarrow$ $\frac{X}{R}$ denote the mapping given by π(x) = class x = [x]. Then π is continuous, open and onto. Let x$\in$X and let $\bar{U}$ be an open set in $\frac{X}{R}$ such that class x $\in$ $\bar{U}$. Then $\bar{U}$ =π(U), for some open set U in X such that X$\in\cup$. Since X is locally connected there exists a connected open set U' in X such that x$\in$U' and U$'$ $\subseteq$ U. Then π(U') is connected and [x]$\in\pi$ (U')$\subseteq$ $\bar{U}$. Hence $\frac{X}{R}$ is locally connected.

Theorem 2

Let X and Y be topological spaces and let F and F$'$ be closed sets of X and Y respectively such that F is homeomorphic to F$'$ are locally connected then X$\#_F$Y is locally connected if both X-int (F) and Y-int (F$'$) are locally connected.

Proof

Let X-int(F) and Y-int (F$'$) be locally connected . Then by Lemma 2, (X-int(F)) + (Y-int(F$'$)) is locally connected . By Lemma 3 and the definition of X$\#_F$Y it follows that X$\#_F$Y is locally connected .

Theorem 3

If both X and Y are compact, then X $\#_F$Y is compact. If X$\#_F$Y is compact then both X-int (F) and Y-int(F$'$) are compact.

Proof

Let both X and Y be compact. We regard X $\#_F$Y as sum (X-int F) + (Y-int F). X-int(F) and Y-int(F) being closed subsets of compact spaces X and Y respectively, are themselves compact. If $\{G_\alpha\}$ is an open cover of X $\#_F$Y, then $\{G_\alpha\}=\{G_\beta\}\cup\{G_\gamma\}$ where $\{G_\beta\}$is an open cover of X-int F and $\{G_\gamma\}$ is an open cover of Y-int(F). Hence for some positive integers m and n, there exists $G_{\beta_1}\ldots G_{\beta_m}$ and $G_{\gamma_1}\ldots G_{\gamma_n}$ such that X-int F $\subseteq$ $G_{\beta_1}\ldots G_{\beta_m}$ and $G_{\gamma_1}\ldots G_{\gamma_n}$ and Y-inf F $\subseteq$ $G_{\beta_1}\ldots G_{\beta_m}$ and $G_{\gamma_1}\ldots G_{\gamma_n}$. Hence $G_{\beta_1}\ldots G_{\beta_m}$ and $G_{\gamma_1}\ldots G_{\gamma_n}$ is a finite subcover of $\{G_\alpha\}$. Therefore X $\#_F$Y is compact.

Now, let X $\#_F$Y be compact. Let $\{G_\alpha\}$, $\{G'_\beta\}$ be open cover of (X-int(F)) and (Y-int (F')) respectively. Let G_α and G'_β be the images of G_α and G'_β in X $\#_F$Y under the identification .

We next prove:

Lemma 4

Let X be a locally compact topological space and Y a closed subspace of X. Then Y is locally compact.

Proof

Let y $\in$Y. Since y$\in$X and X is is locally compact there exists V open in X such that y$\in$V, and $\overline{V}$ is compact in X. Then V$\cap$Y is open in Y and y$\in$V$\cap$Y. Let $\{W_\alpha\}$ be an open cover of $(V\cap Y)_Y$ in Y. Then $\forall_\alpha$ W_α = $U_\alpha\cap Y$, for some U_α open in X. Thus $\{U_\alpha\}$ is an open were of $\overline{(V\cap Y)_Y}$ in X. Since $\overline{(V\cap Y)_X}$ is a closed subset of $\overline{V}$ in X, $\overline{(V\cap Y)_X}$ is compact in X. Hence $\exists$ U_{α_1}, U_{α_2}, ... U_{α_n} such $\overline{(V\cap Y)_Y}\subseteq U_{\alpha_1}\cup U_{\alpha_2}\cup$ .... $\cup U_{\alpha_n}$. So $\overline{(V\cap Y)_Y}\subseteq W_{\alpha_1}\cup W_{\alpha_2}$ ...... $\cup W_{\alpha_n}$. Therefore, Y is locally compact.

We note that X # Y = (X-Y) ∪ (Y-X) ∪C, and that X,Y,C, and Y-X are locally compact by the above lemma, since these are closed subspaces of the locally compact spaces X and Y. Hence $X\#_F Y$ is locally compact here C=b(F).

Lemma 5

If X is locally compact ,then every quotient space of X is locally compact .

Proof

Let X be locally compact and let R be an equivalence relation .Let $z \in \frac{X}{R}$ and let z = cls x , x ∈X . Since X is locally compact , there exists an open set G in X such that x ∈X and $\bar{G}$ is compact. Let G^* is open in $\frac{X}{R}$. Let $\{H_\alpha^*\}$ be an open cover of $\bar{G}^*$ in $\frac{X}{R}$. Let H_α denote the inverse image of H_α^* in X. Then $\{H_\alpha\}$ is an open cover of $\bar{G}$. Since $\bar{G}$ is compact , $\bar{G} \subseteq H_{\alpha_1} \ldots \cup .. \cup H_{\alpha_n}$. Thus $\bar{G}^*$ is compact . So $\frac{X}{R}$ is locally compact .

Lemma-6

If X and Y are locally compact disjoint topological spaces , then X⊕Y is locally compact, and conversely.

Proof

Let X and Y be locally compact disjoint topological spaces. Let Z∈X⊕Y. Then z ∈X or Y . Suppose z ∈ X. Since X is locally compact, there is an open subset G of X such that $\bar{G}_X$ is compact in X. There $\bar{G}_X$

is the closure of G in x. Since $X \cap Y = \phi$, $\overline{G}_X = \overline{G}_{X+Y} = \overline{G}$ say. Let $\{W_\alpha\}$ be an open cover of $\overline{G}$. X⊕Y. Let $\cup_\alpha = W_\alpha \cap X$. Then $\{\cup_\alpha\}$ is an open cover of $\overline{G}$. $\overline{G}$ being compact in X there exists $V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$. So $\overline{G}$. C $W_{\alpha_1} \ldots \cup .. \cup W_{\alpha_n}$. Hence $\overline{G} = \overline{G}_{X\oplus Y}$ is compact in X⊕Y. Thus X⊕Y is locally compact. Since X and Y are closed subsets of X⊕Y both X and Y are locally compact.

Theorem 4

Let X and Y be disjoint topological spaces F, F′, then $X \#_F Y$ is locally compact if and only if X-F, Y-F′ and b(F) are locally compact.

Proof

First X and Y be locally compact. We identify $X \#_F Y$ with (X-F) ∪ b(F) ∪ (Y-F) =(X-Y) ∪ b(F) ∪ (Y-X)= (X-Y) ⊕ b(F) ⊕(Y-Y) .Since (X-Y) , (Y-X) and b(F) are closed subsets of X and Y, Lemma 4 shows that they are locally compact . Hence, by Lemma 6, $X\#_F Y$ is compact.

Theorem 5

Let X and Y be disjoint spaces with homeomorphic closed subsets F and F′ respectively. If X and Y are Hausdorff then $X\#_F Y$ is Hausdorff. It $X\#_F Y$ is Hausdorff, then X-int(F) and Y-int(F′) are Hausdorff.

Proof

Let X and Y be Hausdorff. Using the homoeomorphism of F onto F′ we may identify $X\#_F Y$ with (X-int (F)) ∪ (Y-int (F))’ = (X-F) ∪ (Y-F) ∪b(F) = (X-F) ⊕ (Y-F) ⊕ b (F). Let $z_1, z_2 \in X\#_F Y$. If $z_1, z_2 \in X$, then there exists, disjoint open sets G_1, G_2 in X such that $z_1 \in G_1$, $z_2 \in G_2$. Let $G_1 = G_1 \cap$ (X-int (F)), $G^*_2 = G_2 \cap$ (Y-int (F)).

Then G_1^*, G_2^* are disjoint open sets in X-int(F) and Y-int(F) and $z_1 \in G_1^*$ $z_2 \in G_2^*$. Let $\overline{U}_1$, $\overline{U}_2$ be the images of G_1^*, G_2^* in $X\#_F Y$ and contain z_1, z_2 respectively. Hence is Hausdorff.

Now, Let $X\#_F Y$ be Hausdorff. We write $X\#_F Y$ = (X-F) $\cup$ (Y-F) $\cup$ b(F). Let z_1, z_2 $\in$X-int(F). Regarding z_1, z_2 as elements of $X\#_F Y$, we can find disjoint open sets G_1, G_2 in $X\#_F Y$ which contain z_1 and z_2 respectively. G_1, G_2, the inverse images of G_1 and G_2 are open in X-int (F) and Y-int (F′) respectively. Let $\overline{G}_1$, $\overline{G}_2$ are disjoint open set in X-int (F), and also $z_1 \in \overline{G}_1$, $z_2 \in \overline{G}_2$. Hence X-int (F) is Hausdorff. Similarly Y-int(F′) is Hausdorff.

Comment

The theorem is still valid if 'Hausdorff' is replaced by either 'normal' or 'regular'. The proofs are similar.

We recall that a metric space X is called a **Peano space** if X is compact , connected and locally connected []. Every Peano space is path-connected , and X is a Peano space if X is a Hausdorff and is a continuous image of the closed unit interval I=[0, 1] (see Simmons [15].

Theorem 6

Let X and Y be two metric spaces and let F and F′ two closed subsets of X and Y such that F and F′ are isometric. Then $X\#_F Y$ is a Peano space if and only if X-int(F) and Y- int (F′) are Peano spaces.

First suppose X-int(F) and Y-int(F′) be Peano spaces . Then, X-int (F) and Y-int (F′) are connected, compact and locally . We regard X-int(F) and Y-int(F′) are connected, when F and F′ are identified via the homoeomorphism. So, (X-int(F)) + (Y-int (F′)) connected, since (X-int(F)) ∩ (Y-int (F′)) =b (F) ≠ ϕ. As a quotient space of (X- int (F)) + (Y-int(F′)) , $X\#_F Y$ is conceded. Now let $\{V_\alpha\}$ and $\{W_\beta\}$ be open covers of X-int(F) and Y-int(F′) in $X\#_F Y$. Then $\{V_\alpha\} \cup \{W_\beta\}$ is an open cover of $X\#_F Y$, Since X-int(F) and (Y-int(F′)) and compact, $\{V_\alpha\}$ and $\{W_\beta\}$ have finite subcovers $\{V_{\alpha_1},....V_{\alpha_m}\}$ and $\{W_{\beta_1},....W_{\beta_n}\}$. Then $\{V_{\alpha_1},....V_{\alpha_m}, W_{\beta_1},....W_{\beta_n}\}$ is a finite subcover of $\{V_\alpha\} \cup \{W_\beta\}$ for $X\#_F Y$,so that $X\#_F Y$ is compact .By Theorem 2, $X\#_F Y$ is locally connected. Hence $X\#_F Y$ is a Peano space.

Conversely, lel $X\#_F Y$ be a Peano space. By Theorem 3, both X-int (F) and (Y-int(F′)) are compact. Since $X\#_F Y$ is identified with (X-int(F)) + (Y-int(F′)), Lemma 1 shows that both (X-int(F)) and (Y-int(F)) (and hence (Y-int(F′))) are connected. Lemma similarly proves that X-int(F) and Y-int(F′) are locally connected. Hence both X-int(F) and Y-int(F′) are Peano spaces.

The proof is thus complete.

5. External Sum

From two topological spaces X and Y, we shall now from a third space in a manner similar to the construction of a free product of groups, with amalgamation. This essentially consists in gluing homeomorphic closed subspace of the two spaces.

Definition

Let X and Y be two disjoint topological spaces such that there are non-empty closed subspaces F and F′ of X and Y respectively where F and F′ are homeomorphic to each other. Let f : F → F′ be a homeomorphism. Define a relation R on X ∪ Y as follows :

(i) for each z ∈ (X −F) ∪(Y - F′), zRz′, if and only if z = z′.

(ii) for each x ∈ F, xRx, and xRf(x).

(iii) for each y ∈ F′, yRy, and $yRf^{-1}(y)$.

Consider X ∪ Y as the sum X + Y .Then $\frac{X+Y}{R}$ with the quotient topology will be called the **external sum** of X and Y and will be denoted by

$X \oplus_F Y$ or $X \oplus_{F'} Y$. Clearly ,with the identification via the homeomorphism f , $X \oplus_F Y = (X-F) \oplus F \oplus (Y-F)$.

If X and Y are disjoint subspaces of a topological space then we consider the usual or normal sum of X and Y ,i.e., X ∪ Y with the subspace topology induced topology by Z, and then consider its quotient space modulo R. The resulting space will be called the **usual** or **normal external sum**.

We now study a few properties of the external sum.

Theorem

$X \oplus_F Y$ is connected if and only if both X and Y and connected.

Proof

First suppose X and Y are connected . Since $X \oplus_F Y$ may be looked upon (with proper identification) as $X \cup Y$ with $X \cap Y = F \neq \phi$, it follows that $X \oplus_F Y$ is connected.

Conversely, suppose $X \oplus_F Y$ is connected. If possible, let X or Y, say X, be disconnected. Then there and disjoint open sets G and H is X such that $X = G \cup H$. Then, with identification via the homeomorphism $f : F \rightarrow F'$, $X \oplus_F Y = X \cup Y = X \cup (Y\text{-}F) = G \cup H \cup (Y\text{-}F)$. Here G, H are open sets in X and Y-F is an open set in Y. So by the definition of the topology of a sum and the quotient topology, each of G, H and Y-F is open in $X \oplus_F Y$ and these are pair-wise disjoint. Hence $X \oplus_F Y$ is disconnected, The contradiction proves that both X and Y are connected.

Theorem

$X \oplus_F Y$ s locally connected if and only if both X and Y are locally connected.

Proof

Let $X \oplus_F Y$ be locally connected and let $x \in G$ where G is open in X. Since under proper identification, $X \oplus_F Y = X \cup Y = X + Y$, G is open in $X \oplus_F Y$. So, there exists a connected open set H in $X \oplus_F Y$ such that $x \in H \subseteq G$ .Since $G \subseteq H$, $H \subseteq X$. Thus, H is a connected open subset of X. Hence X is locally connected. Similarly, Y is locally connected.

Now suppose both X and Y are locally connected. Let $z \in X \oplus_F Y$. Regarding $X \oplus_F Y$ as $X \cup Y$, we see that $z \in X$ or $z \in Y$, say $z \in X$. Let G be an open set in $X \oplus_F Y$ such that $z \in G$. Let $G' = G \cap X$. Then , $x \in G'$

and G' is open in X . Since X is locally connected , there is a connected open set H' in X such that $x \in H' \subseteq G'$ .Now H' is also open in $X \oplus_F Y = X \cup Y$, and also, H' is connected as a subset of $X \oplus_F Y$, since ,a disconnection of H' in $X \oplus_F Y$ automatically yields a disconnection of H' in X . Hence $X \oplus_F Y$ is locally connected.

Our next result is

Theorem

$X \oplus_F Y$ is compact if and only if both X and Y are compact.

Proof

Let $X \oplus_F Y$ be compact. Let $\{G_\alpha\}$ be an open cover of X . If we regard $X \oplus_F Y$ as $X \cup Y$ with proper identification ,then $\{G_\alpha\} \cup \{Y\}$ is an open $X \oplus_F Y$. Since $X \oplus_F Y$ is compact for some n , there exist $G_{\alpha_1}, \ldots\ldots G_{\alpha_n}$ such that $G_{\alpha_1} \cup \ldots\ldots \cup G_{\alpha_n} \cup X = X \oplus_F Y$. Hence $X = G_{\alpha_1} \cup \ldots\ldots \cup G_{\alpha_n} \cup X$,so that X is compact . Similarly, Y is compact .

Conversely, let both X and Y be compact Let $\{G_\alpha\}$ be an open cover of $X \oplus_F Y$. We write $X \oplus_F Y = X \cup Y$ and let $H_\alpha = G_\alpha \cap X$ and $L_\beta = G_\beta \cap Y$ Then $\{H_\alpha\}$ and $\{L_\beta\}$ are open cover X and Y respectively. Since X and Y are compact, there exist finite subcover $\{H_{\alpha_1}, \ldots\ldots H_{\alpha_m}\}$ and $\{L_{\beta_1} \ldots\ldots\ldots\ldots L_{\beta_n}\}$ of $\{H_\alpha\}$ and $\{L_\beta\}$ respectively. Then $\{G_{\alpha_1} \cup \ldots\ldots \cup G_{\alpha_m} \cup G_{\beta_1} \cup \quad \cup G_{\beta_n}\}$ is a finite subcover of $\{G_\alpha\}$. Hence $X \oplus_F Y$ is compact .

Theorem

Let X and Y be two compatible topological spaces. If Z is a topological space $f : X \to Z$ and $g : Y \to Z$ are continuous maps such that $f(t) = g(t)$, for each $t \in X \cap Y$. Then $g : X+Y \to Z$ given by $h(x) = f(x)$ for each $x \in X$ and $h(y) = g(y)$ for each $y \in Y$, is continuous .

Proof

Since X and Y are both open subsets of $X + Y$, the theorem follows from the pasting lemma.

Theorem

Let X and Y be two disjoint topological spaces with F and F′ two homeomorphic closed subspaces of X and Y respectively with $\alpha : F \to F'$ a homeomorphism. Let Z be a topological space and $f : X \to Z$ and: $g : Y \to Z$ be continuous maps.

(i) If $f(t) = g\alpha(t)$, for each $t \in b(F)$, then there is unique continuous map $h : X \#_F Y \to Z$ induced by f and g .

(ii) If $f(t) = g\alpha(t)$, for each $t \in F$, then there is a unique continuous map $h : X \oplus_F Y \to Z$ induced by f and g.

Proof

Both $X \#_F Y$ and $X \oplus_F Y$ are quotient spaces of $X \oplus_F Y$. Therefore (i) and (ii) follow from the previous theorem. The map h in that theorem is easily seen to induce the maps h of (i) and (ii) because or the given conditions.

We recall the following result on continuous functions :

Theorem (The Pasting Lemma) (Theorem 7.3 p 108, [61])

Let $X = A \cup B$, Where A and B are closed (or open) in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$, if $x \in B$.

We used this to prove the following result on sums, connected sums and external sums.

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