Department of Mathematics PhD Thesis

2012

Thermodynamics and Hawking Radiation of Black Holes

Hossain, Md. Ismail

University of Rajshahi

Thermodynamics and Hawking Radiation of Black Holes

THESIS SUBMITTED FOR THE DEGREE

OF

DOCTORS OF PHILOSOPHY

IN

MATHEMATICS

BY

MD. ISMAIL HOSSAIN

B Sc Hon's(1st. class), M Sc (1st class)

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF RAJSHAHI

RAJSHAHI-6205

BANGLADESH

ROLL NO. 09319

REG. NO. 3570

SESSION: 2009-2010

SEPTEMBER- 2012

Dr. M. Abdullah Ansary
Professor
Department of Mathematics
University of Rajshahi.
Rajshahi-6205, Bangladesh.

Residence: 176/Kazla, Ra
Phone:0721-750526
Fax::
Mobile: 01199379144
E-mail:ansary-math@yah

Dated: 29/09/2012

Certificate from the supervisor

This is to certify that the thesis entitled **"Thermodynamics and Hawking Radiation of Black Holes** "submitted by MD. Ismail Hossain who got his name registered in July/2009 M Phil/ Ph D. batch for the award of the degree of Doctor of philosophy in mathematics of Rajshahi University , is absolutely based on original work under my supervision and neither this thesis nor any part of it has been submitted for any degree or any other academic award elsewhere before.

m. a. Ansary

(Dr. M. Abdullah Ansary)
Supervisor

Declaration

I do hereby declare that the thesis entitled **"Thermodynamics and Hawking Radiation of Black Holes"** submitted by me to the University of Rajshahi for the award of Ph D. degree in mathematics has not been submitted to any institute or University for degree or award.

29.09.12

(Md. Ismail Hossain)
Ph D. Fellow
Department of mathematics
University of Rajshahi
Rajshahi-6205, Bangladesh.

Dedicated

To

my parents

Acknowledgement

At first I would like to express my gratefulness to the omnipotent creator and sustainer of the universe for enabling me to complete the work of this thesis.
I am highly thankful and obliged to my esteemed supervisor, my constant source of knowledge Dr. M. Abdullah Ansary , Department of mathematics, University of Rajshahi, for his guidance, encouragement and support throughout the course of this work and in the preparation of this thesis.

I would also like to express my gratitude to the honorable chairman, Department of mathematics, University of Rajshahi, Rajshahi for providing me the departmental facilities. I also wish to record my sincere thanks to all the teachers and staff, Department of mathematics , University of Rajshahi, Rajshahi, for their cooperation and inspiration that I have had during this work.

I would like to take this opportunity to remember all my teachers , from whom I got lessons and inspiration to carry out advanced study.

I also express my profound gratitude to my reverend teacher professor (ret.) K. A. Dakua for his constant encouragement and inspiration and Dr. Sabur Uddin for his valuable advice during this course of work.

I am grateful to the University Grant Commission (UGC) of Bangladesh for awarding a fellowship (vide no. 10/6094 , Date: 27.06.2010, September 2010-September 2012) during this course of work.

I am also grateful to the Ministry of Education, Government of the People's Republic of Bangladesh for giving me kind permission and granting me required study leave/ Deputation for perusing the Ph D. course.

Many thanks go to my wife for her encouragement and support during the entire period of this work.

The author

ABSTRACT

After a short review of spacetime singularities , blackholes , we introduce one with the laws of blackhole mechanics and the laws of ordinary thermodynamics. We discuss the remarkable analogy between the laws of blackhole mechanics and the laws of thermodynamics. By Bekenstein proposal we explain the flaws arises when one attempts to draw an analogy between them. We study the Bekenstein-Hawking entropy, evidence of blackhole entropy, interpretation of blackhole entropy, the linearity of blackhole entropy with its horizon area , the problem of blackhole entropy and using thermodynamic relation we obtain Bekenstein-Hawking entropy, Hawking temperature and some intensive parameters of some different kinds of blackholes.

In this thesis, we also study the Hawking radiation, its nature and a parallel discussion with blackbody radiation. The luminosity and lifetime of blackholes are also studied. By applying Parikh-Wilczek's semi-classical tunneling method we obtain the emission rate of massless uncharged particle and the massive charged particles at the event horizon of blackholes. Finally, we obtain the emission rate at the event horizon of some kinds of blackholes by applying a new method known as Hamilton-Jacobi method.

List of Tables

List of Figures

Contents

CHAPTER THREE:

THERMODYNAMICS AND BLACKHOLE MECHANICS

CHAPTER FOUR:

BLACKHOLE ENTROPY

CHAPTER FIVE:

BLACKHOLE RADIATION AND UNCXHARGED PARTICLES TUNNELING.

CHAPTER SIX:

HAWKING RADIATION AS TUNNELING OF CHARGED PARTICLE

CHAPTER ONE

INTRODUCTION

The most exotic entities encountered in the present study of physics are blackholes. The nature of blackhole spacetime is enough to make the physics of blackholes more than science fiction. In the prologue to the Mathematical Theory of Blackholes Subrahmanyan Chandrasekhar sums up his views on blackholes in a sentence : "The blackholes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time." Even more astounding are the connections of blackhole physics with thermodynamics. One of the most remarkable developments in theoretical physics that has occurred in the past forty years, was undoubtedly the discovery of the close relationship between the certain laws of the ordinary thermodynamics and the laws of blackhole mechanics. The starting point of this remarkable developments was the discovery of the four laws of blackhole mechanics by Bardeen, Carter and Hawking [1]. It appears that the laws of blackhole mechanics and the laws of thermodynamics are two major pieces of a puzzle that fit together so perfectly that there can be little doubt that this 'fit' is of deep significance. The existence of this close relationship between these laws seem to be guiding us towards a deeper understanding of the fundamental nature of spacetime, as well as understanding of some aspects of the nature of thermodynamics itself [2].

It was first pointed out by Bekenstein [3] that a close relationship might exist between the certain laws satisfied by blackholes in classical general relativity and the ordinary laws of thermodynamics. He noted that the area theorem of classical general relativity is closely analogous to the statement of the ordinary second law of thermodynamics. His proposal was confirmed by Bardeen, Carter and Hawking[1], they proved that in general relativity, the surface gravity , κ, of a stationary blackhole must be constant over the event horizon, which is analogous to the zeroth laws of thermodynamics. The analogue of the first law of thermodynamics was also proved.

It is generally believed that classically a blackhole is nothing but a perfectly dead star which have an absolute zero as a physical temperature. But it was not so since Hawking has found a startling discovery that the blackholes radiates thermally[4], whereas Bekenstein suggested that there is an entropy associate with the blackhole [5]. However that the blackhole has an entropy first arose from the realization that its event horizon surface area exhibits remarkable tendency to increase when undergoing any transformation as noticed by Floyd and Penrose[6] and later supported by Christodoulou [7]. Hawking [8] was the first to give a general proof that the surface area of the blackhole cannot decrease in any process and additionally he showed that when two blackholes coalesce, the area of the resulting blackhole cannot be smaller than the sum of the initial areas. It is clear that the change in blackhole generally occur in the direction of increasing area. This is reminiscent of the second law of thermodynamics which states that the changes of a closed thermodynamic system takes place in the direction of increasing entropy. This comparison suggests that it might be useful to consider blackhole physics from thermodynamic viewpoint, that something like entropy may play a major role in it. However, physicist were not convinced about the validity of blackhole entropy before Hawking radiation was discovered.

An incredible outcome of the Einstein theory of gravity are blackholes. They were thought that no matter inside could escape and so invisible from outside. In 1970s, Hawking startled all the physical community by proving that the blackholes are not actually black[4,9]. They can radiate thermally like a blackbody with Hawking temperature $T_H = \frac{\hbar\kappa}{2\pi}$ where κ is the surface gravity of the blackhole. The surface gravity means the acceleration measured at the spatial infinity that a stationary particle should undergo to withstand the gravity at the horizon. Although the heuristic picture which visualizes the source of radiation as tunneling was first proposed by Hawking, but his calculation was completely based on quantum field theory in curved spacetime which is independent of a tunneling process.

The classical 'no hair' theorem stated that all the information about the collapsing body was lost except three conserved quantity: the mass, the angular momentum and the electric charge. So the only solutions of Einstein-Maxwell equations in four dimensions is the stationary and rotating Kerr-Newman blackhole solutions. In classical theory , the loss of information is not a serious problem since it could be thought that the information is preserved inside the blackhole but just not very accessible. Even , once Hawking thought that the loss of information never recovered.

But recently he change his opinion about information loss paradox. However , taking quantum effect into consideration , the situation is changed due to Hawking discovery that blackholes radiates thermally[4,9]

Due to the emission of thermal radiation blackhole could loss energy, shrink and eventually evaporate away completely. Since the radiation with a precisely thermal spectrum carries no information , so the information carried by a physical system falling toward blackhole singularity has no away to be recovered after a blackhole has disappeared completely. This is known as so called " information loss paradox"[10] which means that pure quantum states (the original matter that forms the blackhole) can evolve into mixed states (the thermal spectrum at infinity). This type of evolution violates the fundamental principle of quantum theory, as these prescribe a unitary time evolution of basis states[11].

The information loss paradox can perhaps be attributed to the semi-classical nature of the investigations of Hawking radiation. However, researches in string theory indeed support the idea that Hawking radiation can be described within a manifestly unitary theory, but it still remains a mystery how information is recovered. Although a complete resolution of the information loss paradox might be within a unitary theory of quantum gravity or string/ M-theory , it is argued that the information could come out if the outgoing radiation were not exactly thermal but had subtle corrections[10].

After Hawking's discovery that blackholes radiate[4,9], there were several approaches to study this effect. The Hawking discovery was based on the general relativity and quantum mechanics. This is the key link in spacetime quantization. In the last few decades , there were many researches on the Hawking radiation and many methods to calculate Hawking radiation were obtained.

There is some degree of mystery remains in the mechanism of blackhole radiation. In the original derivation of blackhole evaporations, Hawking described the thermal radiation as a quantum tunneling process created by vacuum fluctuation near the event horizon [12]. In this process , the radiation is like electron-positron pair creation in a constant electric field. The energy of a particle can change its sign after crossing the event horizon. So a pair created by vacuum fluctuations just inside or outside the horizon can materialize with zero total energy, after one member of the pair has tunneled to the opposite side. But in [4] Hawking did not proceed in this way. He considered the creation of a blackhole in the context of a collapse geometry, calculating the Bogoliubov transformations between the initial and final states of incoming and outgoing radiation. However , there were two difficulties to overcome this problem. The first was to find a well – behaved coordinate system at the event horizon. The second was where is the barrier.

Recently , a method to describe Hawking radiation as tunneling process was developed by Krause and Wilczek [13] and elaborated by Parikh and Wilczek[14,15,16,17]. It was suggested in the method that the barrier is created by the tunneling particle itself. This method involves calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon, which in turns is related to the Boltzmann factor for emission at the Hawking temperature. Using the (Wentzel- Kramers –Brillouin) WKB approximation[1] the tunneling probability for the classically forbidden trajectory of the s-wave coming from inside to outside the horizon is given by

$\Gamma \propto \exp(-2\,\mathrm{Im}\,S)$

where S is the classical action of the trajectory to leading order in $\hbar$(set equal to unity).

Expanding the action in terms of the particle energy , the Hawking temperature is recovered at linear order. In other words for

$2S = \beta E + 0(E^2)$ this gives

$\Gamma \sim \exp(-2S) \cong \exp(-\beta E)$

which is the regular Boltzmann factor for a particle of energy E and β is the inverse temperature of the horizon.

Besides treating the Hawking radiation as a tunneling process Krause-Parikh-Wilczek also took the tunneling particles back reaction into account. They obtained the corresponding modified spectrum.

[[1] For large values of the quantum numbers or of the masses of the particles in the system the quantum mechanics gives results closely similar to classical mechanics. For intermediate cases it is found that the old quantum theory often gives good results. It is therefore pleasing that there has been obtained an approximation method of solution of the wave equation based on an expansion the first term of which leads to the classical result , the second term to the old-quantum theory result , and the higher terms to corrections which bring in the effects characteristic of the new mechanics. This method is usually called the Wentzel-Kramers-Brillouin method (precisely the WKB approximation method)]

The most interesting result was that they found this modified spectrum was implicitly consistent with the unitary theory and could support the conservation of information[13,14,15,16].

Following this tunneling method , there have been many generalizations , such as its application to other spacetimes. The Hawking radiation as tunneling from various spherically symmetric blackholes were found in [11,18,19,20,21,22,23,24,25,26,27,28,29,30].There are some attempts to extend this method to the case of stationary axisymmetric blackholes [31,32,33,34,35,36,37,38,39]. Recently, some researchers investigated the massive charged particles tunneling from the static spherically symmetric as well as stationary axisymmetric blackholes [40,41,42,43,44,45,46]. They all found a satisfying result. However , Parikh and Wilczek's tunneling method is dependent on coordinates, which means that it should find a Painleve-like coordinates. There is a new method which is independent of coordinates and known as Hamilton-Jacobi tunneling method developed by Angheben, Nadalini,Vanzo and Zerbini[31]. This variant tunneling method could also be considered as an extension of the method used by Padmanabhan ,Srinivisan, Shankaranarayann and Vegenas [47,48,49,50,51]. More research paper in this area are also found [52,53].

In this thesis we review spacetime singularity, the blackholes and their formation, some classification and some properties in chapter two. Some established theorems on blackholes and present observational evidence are also added in this chapter.

In chapter three, introducing one with the laws of blackhole mechanics and the laws of ordinary thermodynamics ,we briefly review the remarkable analogy between ordinary thermodynamics and blackhole mechanics. We also discuss the validity and necessity of the generalized second law (GSL). We explain the flaws arises when one attempt to draw an analogy between the laws of blackhole mechanics and the laws of ordinary thermodynamics.

In chapter four, we give some evidence of blackhole entropy, blackhole entropy expression, the linearity of blackhole entropy with its horizon area and some interpretation of blackhole entropy given by the various researchers. We also discuss the blackhole entropy problem and sums up some open questions to which complete answers to these questions is still lack. Finally, using thermodynamic relation we obtain Bekenstein-Hawking entropy, Hawking temperature and some other intensive parameters of various types of blackholes.

In chapter five, we give a short history of Hawking radiation, the nature of Hawking radiation , either the Hawking radiation is continuous or discrete and given a parallel discussion between blackhole radiation and blackbody radiation. The luminosity and lifetime of blackholes are also discussed in this chapter. The tunneling of uncharged massless particles of various types of blackholes are also given in this chapter and we obtain the tunneling probability of some blackholes.

In chapter six, we discussed the tunneling probability of massive charged particles which are obtained by the some researchers. Following their methods and techniques, we obtained the tunneling probabilty of massive charged particles from some kinds of blackholes.

In chapter seven, a new method to study the Hawking radiation as tunneling the Hamilton-Jacobi methods are discussed. In this chapter, applying this method we obtain the tunneling probability of some blackholes.

CHAPTER TWO

SPACETIME SINGULARITY

2.1 Singularities:

Mathematically, a singularity of a function is a condition when the function does not give a finite value. For example, in Newtonian mechanics the gravitational potential energy U of a mass m is given by the equation $U = -\frac{GMm}{r}$, where G is the Newton's gravitational constant, M is the mass of attracting body and r is the distance between the two centers of the bodies. Here U becomes infinite when $r=0$, therefore $r=0$ is a singularity of U. In the context of general relativity theory, spacetime singularity means the region or location of the space in which the Einstein field equations break down. Einstein field equation are taken to be a fundamental description of space and time. At the singularity, objects or light can reach a finite time but the curvature of spacetime becomes infinte. Singularity lies inside the blackhole where matter is crushed in infinite density, the pull of gravity is infinitely strong and spacetime has infinite curvature. In the solution of Einstein equations, a situation where matter is forced to be compressed to a point is called a spacelike singularity and a situation where certain light rays come from a region with infinite curvature is called timelike singularity.

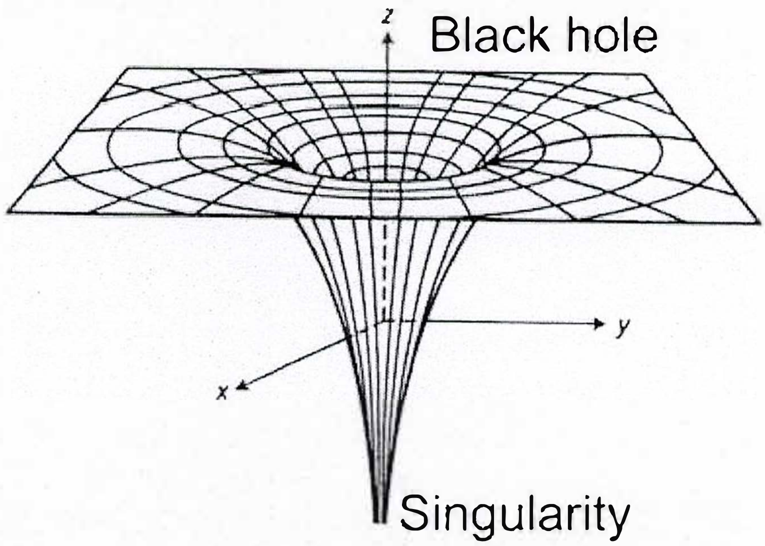

Figure: 2.1 Formation of singularity.

Spacelike singularities are a feature of non-rotating uncharged blackholes, while time like singularities are those that occur in charged or rotating blackhole exact solution

Within a few months after Einstein field equations discovered, Karl Schwarzschild obtain the solution of Einstein equation, for vacuum space, $R_{\mu\nu} = 0$ as

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad \text{.............................(2.1)}$$

(with $G = c = 1$)

Here M is the mass of the matter, r is the distance from the center of the matter. The equation (2.1) has a singularity at $r = 0$ and $r = 2M$ .The singularity at $r = 0$ is a true singularity or physical singularity since it cannot remove by any co-ordinate choice. But the singularity at $r = 2M$ is not a true singularity since it can be removed by a suitable co-ordinate choice. In ingoing Eddington-Finkelstein(EF) coordinate system (v, r, θ, ϕ) where $v = t + r_*$ with r_* is defined as

$$r_* = \int \frac{1}{1 - \frac{2M}{r}} dr = r + 2M \ln(\frac{r}{2M} - 1) \qquad \text{......................................(2.2)}$$

In this coordinate system the metric (2.1) takes the form

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad \text{........................(2.3)}$$

and we see that there is no singularity at $r = 2M$. Thus we have two characterizations of spacetime singularity in Schwarzschild solution (i) a singularity that cannot be removed by any choice of coordinate and (ii) the singularity which can be removed by a suitable coordinate choice, while these criteria work for blackholes, however, they are not sufficient to capture all spacetime singularities.

The metric (2.3) defined for $r > 2M$ since the relation $v = t + r_*$ between v and r is only defined for $r > 2M$, but it can now be analytically continued to all $r > 0$.Because of the $drdv$ cross term the metric in EF coordinate is nonsingular at $r = 2M$, so the singularity in Schwarzschild coordinates was really a coordinate singularity. There is nothing at $r = 2M$ to prevent the star collapsing through $r = 2M$. This is illustrated by a Finkelstein diagram, which is a plot of $t_* = v - r$ against r.

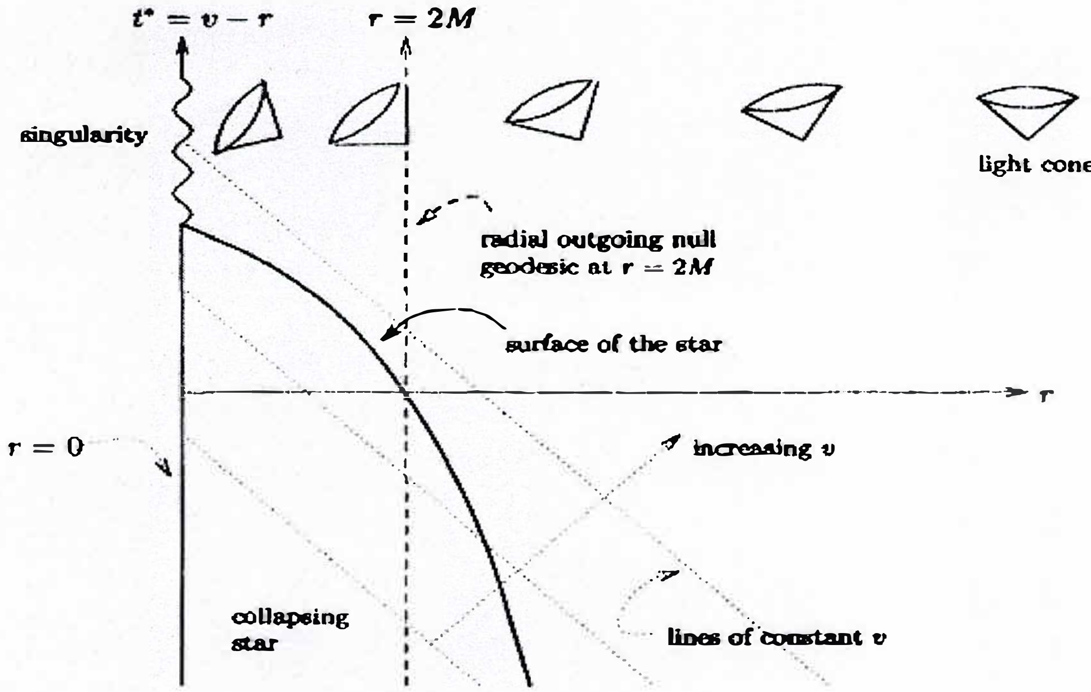

Figure:2.2

The light cones distorted as $r \rightarrow 2M$ from $r > 2M$ so that no future directed timelike or null world line can reach $r > 2M$ from $r \leq 2M$ [54].

Spacetime singularities are also explained by geodesics. Geodesics are the 'possible straightest' path of spacetime. For any geodesics we can extend it infinitely on both sides. If this is not possible then it seems that the geodesic path comes to an edge or an end in some finite distance. Therefore we give a characterization of spacetime singularity in terms of "geodesic incompleteness". A spacetime is called singular if it contains geodesics that cannot be extended to infinity. In this case it seems that there is an 'edge' or an 'end' to spacetime which lies at finite distance. For blackholes it can be shown that the geodesic paths can be extended through $r = 2M$ but not $r = 0$.

2.2 Blackhole:

In the realm of science ,blackholes were at first only a speculation as a result of calculation of the bodies whose escape velocity is greater than the velocity of light. At first , in 1784, John Michell gave this idea. After a few years, in 1798, mathematician Pierre Simon Laplace discussed about the classical bodies with escape velocity greater than the speed of light. But at that time their idea could not attract much attention. After discovery of Einstein's theory of general relativity, the theoretical discussion about that bodies again started. In 1967, John Wheeler, an American physicist coined the term 'blackhole' and thereafter it is popular used.

[John Wheeler always denied that he coined the term 'blackhole'. He says that ,in the fall of 1967 ,he was invited to give a talk on pulsars, then mysterious deep space object at NASA's Goddard institute of space studies in New York. As he spoke, he argued that something strange might be at the center ,what he called a gravitationally completely collapsed object. But such a phrase was a mouthful, he said wishing about for a better name. 'How about blackhole?' some one shouted from the audience.

That was it I had been searching for just the right term for months, mulling it over in bed, in the bathtub, in my car wherever I had quiet moments, he later said. Suddenly this name seemed exactly right. He kept using the term, in lectures and on papers and it stuck.]

The simplest picture of blackhole is that of a body whose gravity is so strong that nothing , even light cannot escape from it. The escape velocity of a body means the initial speed that required to go from an initial point in a gravitational potential field to escape the gravitational pull of the body and continue flying out to infinity. For example, the escape velocity of the earth is 11.2 km/s and for the moon it is 2.4 km/s. According to the theory of relativity, nothing can propagate faster than the speed of light and so if light cannot escape due to strong gravity of the body, then neither can anything else. So the body is unobservable and treated as a blackhole.

2.3 Event horizon of blackhole :

The important key to understanding the study of blackholes is event horizon. Simply ,horizon is a boundary in spacetime in which matter and light can only goes to inward towards the center of the blackhole. In this sense ,the event horizon is a place of no return. More generally, horizon means the boundary between the part of spacetime from which light can escape to infinity and the part out of which light cannot escape. So it is separating the events from outside universe. Within the boundary if an event occurs, the information from that event cannot reach outside observer. For a distant observer clocks near a blackhole appear to tick more slow down than those further away from the blackhole. This effect is known as gravitational time dilation. If an object approach the event horizon and cross it, then for a distant observer it would like to move slower and slower as it closer and closer to the horizon. Observer seems that the object never reach at the horizon though the falling objects pass through the horizon in a finite amount of proper time. For a non rotating , uncharged Schwarzschild blackhole the spherical surface is referred to an event horizon while for rotating blackholes, event horizons are distorted non spherical.

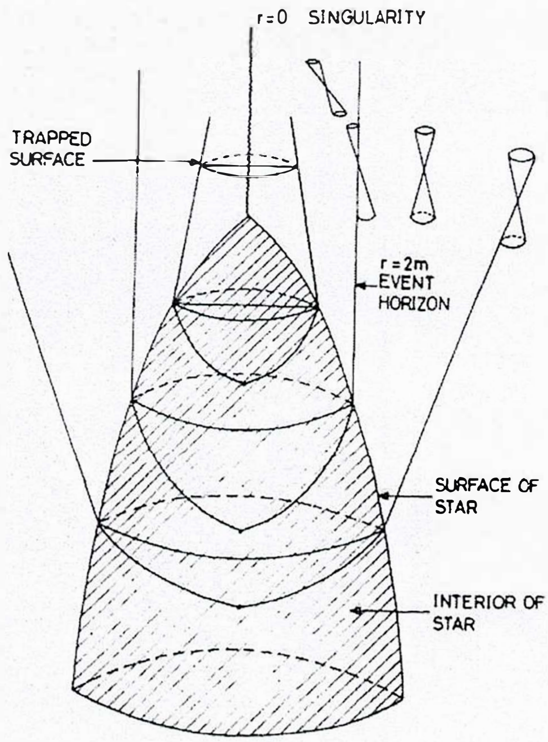

Figure: 2.3 The spherically symmetric collapse of a star, showing the formation of an event horizon that is the boundary of the reign of space-time from which it is not possible to escape to infinity. In this diagram time is plotted vertically and space horizontally, with one spatial dimension suppressed.

D- 3645

In fact the more accurate description of event horizon is that, at a specific distance from blackhole light cones are so tipped over that the outgoing edges of each light cone is vertical in the diagram below.

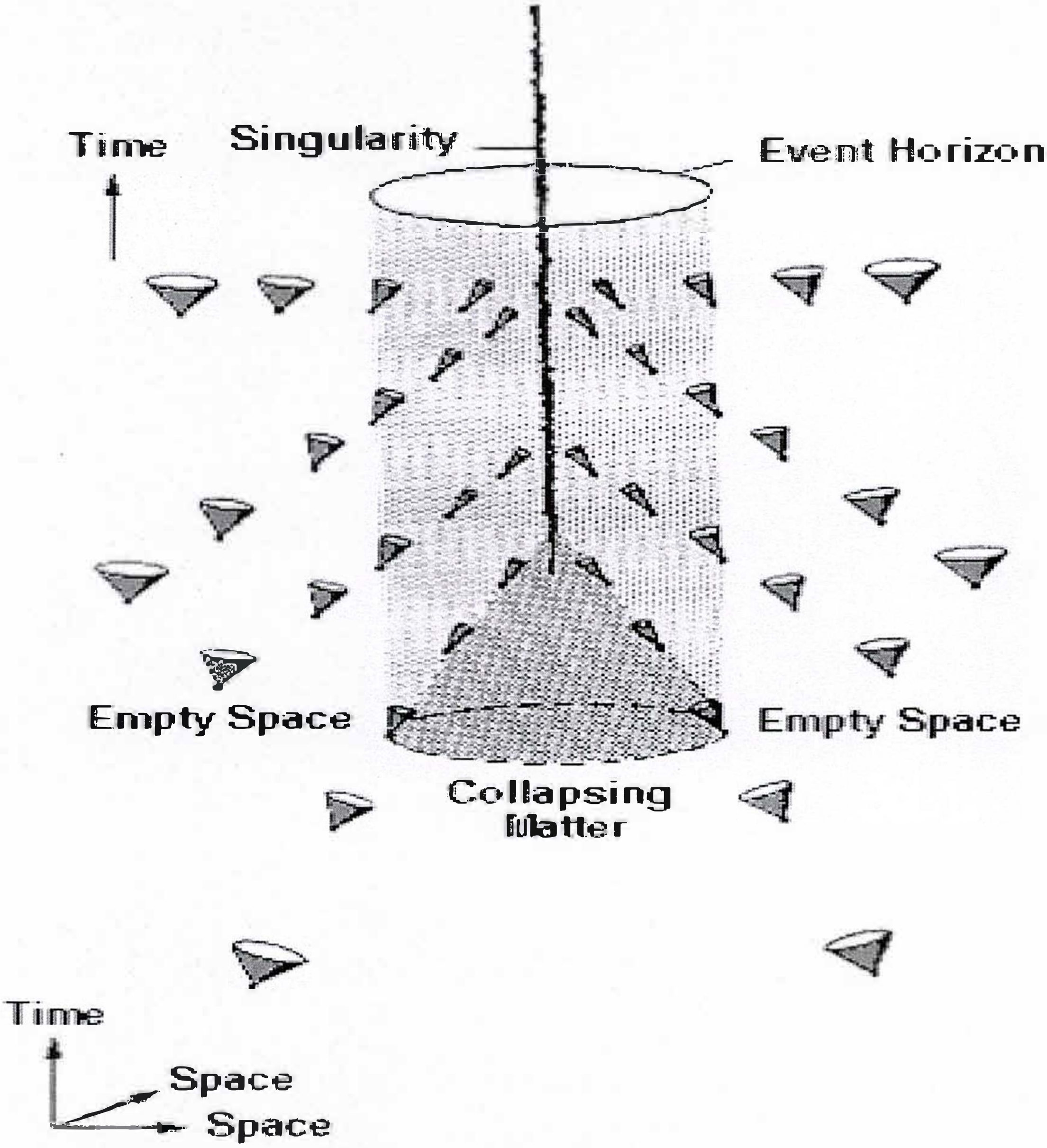

Figure. 2.4 From Penrose (Scientific American).

These edges form a surface which is called the event horizon. The boundary divides the spacetime into an 'out side' and an 'inside' where as from inside particles and light rays can never escape outside because all of the light cones point to the singularity, their world lines will end.

2.4 Formation of blackhole:

The solution of Einstein field equations suggest that such a bizarre objects , like blackholes could exist in nature. But Einstein himself thought that black holes would not form, because he held that the angular momentum of collapsing particles would stabilize their motion at some radius[55].He claimed that the collapsing matter could not reach at zero volume. This led the general relativity community to dismiss all results to the contrary for many years. Only a minority of relativists continued to contend that blackholes were physical objects[56] and by the end of 1960's they infer that there is no obstacle to forming a blackhole in nature.

Consider a very compact and massive star. The strength of gravity of the star can be increased if the star shrink or more mass is added. When light rays leave the surface of this star radially outwards then gravity affects the light due to its particle properties(due to photon mass).To overcome the surface gravity and escape from the star ,light has done some work. So its energy and hence frequency will be diminished. As a result gravitational red shift occur. For more compact and massive star the red shift becomes infinite. For example, if a clock at rest in the metric (2.1) and located at a distance r ($r > r_s$) exhibits, when its ticks are 'read' from infinity via electromagnetic signals a red shift measured by [57]

$$\frac{[ds]^{read\ at}_{\inf inity}}{[ds]^{measured}_{locally}} = \frac{\sqrt{-g_{00}(r=\infty)}}{\sqrt{-g_{00}(r=r)}} = \frac{1}{\sqrt{1-\frac{2M}{r}}} \qquad (2.4)$$

The red shift (2.4) goes to infinity if $r \to r_s$. Here $r_s = 2M$, Schwarzschild radius. To get an idea about Schwarzschild radius it is notable that the Schwarzschild radius of the sun is approximately 3 km and for the earth it is about 1 cm . This means that if we could collapse all the earth's matter down to a sphere whose radius is 1 cm , then it will form a blackhole.

In actual world blackholes may formed by the following process:

2.4(a) Gravitational collapse:

The primary formation process of blackholes is expected to be the gravitational collapse of sufficient amount of matter. When a star consumes its nuclear fuel , then it stops all the thermal activity that prevents it from collapsing under its won weight. Then the star is known as death star and will undergo a gravitational collapse. The collapsed may be stopped by the degeneracy pressure of the stars constituents condensing the matter in an exotic denser state. The fate of the death star depends on the mass of the remnant. In 1931, Subrahmanyan Chandrasekhar calculated if the mass of the remnant less than 1.44 times solar masses (known as Chandrasekhar limit) then electron degeneracy pressure of it prevents itself to collapsing. The star is then stable and known as 'white dwarf '.If the mass of the remnant lies between 1.44 to 3 solar masses ,then the star again collapse and get a size smaller than 'white dwarf '.In this case neutron and proton degeneracy pressure counter balance the gravity of its weight [58].

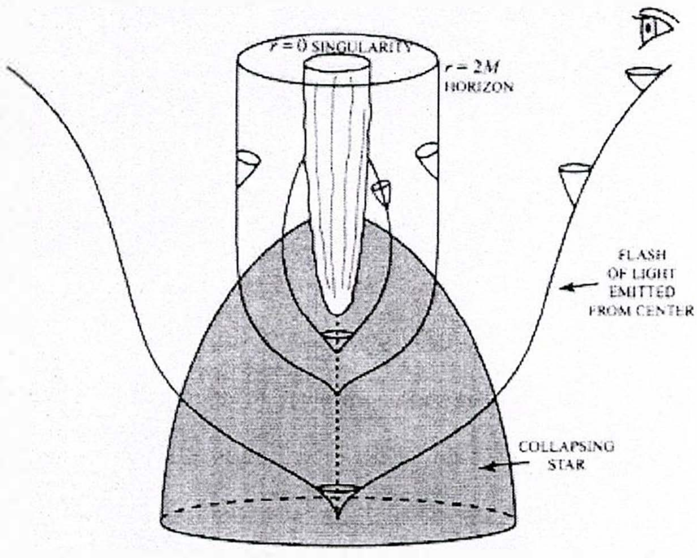

Figure.2.5 Space-time representation of the formation of a blackhole by the collapse of a star.

The star is again stable and is called the 'neutron star'. The radius of the 'neutron star' may be only 10 miles and density per cubic inches billion billion tons[58].Finally if the mass of the remnant exceeds about 3-4 solar masses(Tolman-Oppenheimer-Volkoff limit) then there is no known mechanism is powerful enough to stop the collapse and the star will form a blackhole.

2.4 (b) Collapse of star cluster:

If a galaxy is densely populated with stars then after a long time its center become more and more condensed by the star cluster. This evolution may form a single supermassive body at the center of the galaxy. This supermassive body may then undergo gravitational collapse and form a black hole. Some supermassive blackhole with mass 10^5 to 10^9 solar masses will be form by this process. Also some intermediate blackholes are supposed to be formed by the amalgamation of many smaller and cosmic bodies.

Table-2.1 (various types of blackhole)

class	mass	size
Suppermassive blackhole	$\sim 10^5 - 10^9 M_\otimes$	~.001-10AU
Intermediate blackhole	$\sim 10^3\ M_\otimes$	$\sim 10^3 km = R_{earth}$
Steelar blackhole	$\sim 10 M_\otimes$	~30km
Micro blackhole	Upto $\sim M_{moon}$	Upto ~0.1mm

[From Wikipidia,the free encyclopedia]

2.4 (c) Primordial blackholes:

It is possible that after a very short time of 'Big Bang' densities of matter were very much greater allowing for the creation of blackholes. The high density alone is not enough to allow the formation of blackholes since a uniform mass distribution will not allow the mass to bunch up[58]. In other words if the matter density was enhanced in some region, then rather than expand with the rest of the universe, gravitational collapse of the matter in this region to form a blackholes might have occurred. Stephen Hawking proposed that trillions of non stellar blackholes or primordial blackholes were created along with the universe in accordance with 'Big Bang'. Some body suggest that high energy particle collisions produce the required dense matter that can create a mini blackhole. But in the present universe it is not possible to form such blackholes because gravitational collapse and collapse of a star cluster cannot produce blackholes of very low mass.

2.5 Different types of blackholes:

2.5 (a) Schwarzschild blackhole:

The story of the blackholes begins with Schwarzschild discovery of the Schwarzschild solution in 1916, soon after Einstein formulation of final gravitational field equations in 1915. The Schwarzschild solution is the first simplest exact solution of the vacuum Einstein equations which is spherically symmetric and involving only one parameter M, the mass. This solution or blackhole has no angular momentum, no charge and cannot be distinguished from any other Schwarzschild blackholes except by its mass. The solution is given by the metric (2.1) and has a singularity at $r=0$ and $r=2M$. The singularity at 2M due to its coordinates where the spacetime change their meanings.

We see that the light cones in Schwarzschild coordinates are closing up as we approach $r=2M$. So we can contrast a better coordinate system in that region by following casual structure; define new coordinates

$$u, v = t \pm r_{*} = t \pm [r + 2M \ln(\frac{r}{2M} - 1)] \qquad (2.5)$$

and so $u,\ v = t \pm \dfrac{r}{1-\dfrac{2M}{r}}$ ....(2.6)

Thus ingoing null rays have $u = constant$, while outgoing null rays have $v = constant$. If we write the metric in coordinates (u, r, θ, ϕ) we can extend it across $r=2M$ along ingoing null rays. Similarly the metric in coordinates (v, r, θ, ϕ) can be extended across $r=2M$ along outgoing null rays.

In Kruskal-Szekeres coordinates which are defined by

$$u' = e^{\frac{u}{4M}} = (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{(r+t)}{4M}} \qquad (2.7)$$

$$v' = -e^{-\frac{v}{4M}} = (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{(r-t)}{4M}} \qquad (2.8)$$

In terms of these coordinates , the metric (2.1) becomes

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} du' dv' + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \quad (2.9)$$

Where $r(u,v)$ is defined implicitly by (2.7),(2.8) .These coordinates are maximal- all geodesics either extend to infinite affine parameter without leaving this chart or meet the singularity at $r = 0$ .The singular surface at $r = 2M$ in the previous coordinates maps to $u'v' = 0$ which is manifestly non singular. On the other hand, $r = 0$,which maps to $u'v' = -1$ is still singular; this is a curvature singularity. More generally, surfaces of constant t are at $\frac{u'}{v'} = constant$,while surfaces of constant r are at $u'v' = constant$.

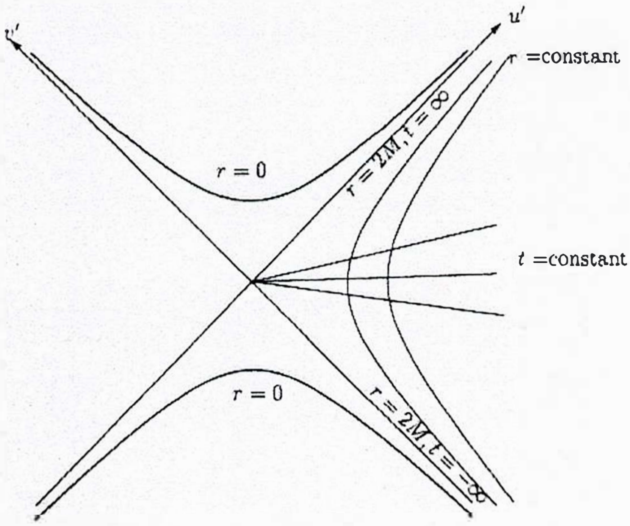

Figure:2.6 Kruskal diagram for Schwarzschild .

For large r , the metric (2.1) takes the form,

$$ds^2 \approx -(1 - \frac{2M}{r}) dt^2 + (1 + \frac{2M}{r}) dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \quad (2.10)$$

and from this equation, one can easily show that the Newtonian gravity is merely a limiting case of general relativity. Again if we take $r \to \infty$ in (2.10) we obtain,

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \quad (2.11)$$

Which is Minkowski flat spacetime.

2.5 (b) Ressiner-Nordstrom blackhole:

Hans Ressiner and Gunnar Nordstrom discovered the solution of Einstein equation for vacuum space for a charged body of mass M. Their solution is given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

....(2.12)

($with\ G = c = 1$)

Here M is the mass of the body and Q is the total charge of the body. The singularities of equation (2.12) is given by

$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$ ....(2.13)

which gives,

$r_{\pm} = M \pm \sqrt{M^2 - Q^2}$, $for\ M^2 > Q^2$ ....(2.14)

Therefore the two concentric event horizons becomes degenerate for $M = |Q|$ which corresponds to an extremal blackhole. The blackhole with $Q > M$ are believed not to exist in nature. It is notable that the charged blackhole may not be observed in nature because the blackhole is already discharged when it is in stable state. The time taken by the blackhole from charged to discharged state called the characteristic time , is approximately $10^{-5}\frac{M}{M_{\otimes}}$ second [59].So the blackhole having mass $10^{-5} M_{\otimes}$ becomes stationary state within one second !

2.5 (c) Kerr blackhole:

In 1963, Roy Kerr obtained a solution of Einstein equation for uncharged rotating body. The solution is given by the metric known as Kerr metric or Kerr blackhole as (in the Boyer-Lindquist coordinates)

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a\,dt]^2$$

.......................(2.15)

(with $G = c = 1$)

where

$\Delta = r^2 - 2Mr + a^2$

$\rho^2 = r^2 + a^2\cos^2\theta$

$a = \frac{J}{M}$

Here M is the mass of the body, J is the angular momentum and r is the radial distance from the center of the body. The Kerr metric is used to describe a rotating blackhole. The singularity of equation (2.15) is given by,

$\Delta = 0$ ....(2.16)

Which gives,

$r_{\pm} = M \pm \sqrt{M^2 - a^2}$, $M^2 > a^2$. ....(2.17)

So we may define three distinct region of the Kerr solution bounded by the event horizons:

Region-1 : $r_{+} < r < \infty$

Region-2 : $r_{-} < r < r_{+}$

Region-3 : $0 < r < r_{-}$

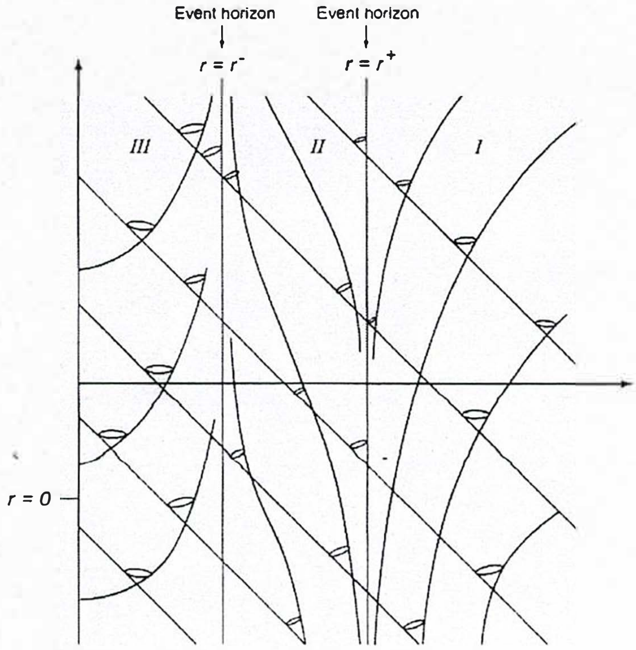

Figure:2.7 Space-time diagram of the Kerr solution in advanced Eddington-Finkelstein coordinates.

The above figure shows that the three regions plotted in a spacetime diagram along the equator of the blackhole using advanced Eddington-Finkelstein (EF) coordinates in which ingoing null rays are straight lines.

Therefore the Kerr blackhole have different two surface where the metric appears have a singularity. The size and shape of these two surface depends on both M and J. The region between outer surface and inner surface is called 'Ergosphere'. The outer surface enclose the ergosphere and its shape is similar to flattened sphere. The inner surface marks the event horizon. Objects which passes through the event horizon can never communicate with the outside universe. Objects which comes close enough to the blackhole so that they enter the ergosphere are

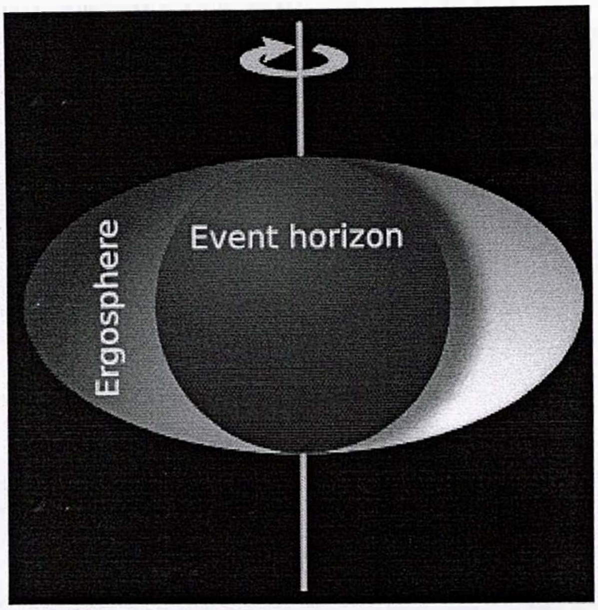

Figure:2.8 Kerr blackhole surrounded by an ergosphere. The ergosphere is a region inside which nothing can remain stationary.

forced to rotate in the same direction as the rotating matters which collapse to form the blackhole. This feature can be used to extract energy from rotating blackhole.[Penrose process] The Kerr blackhole is extremal when $|a| = M$ *i.e.* $J = GM^2$ and if there is no spin i.e. $J = 0$ then it reduces to a Schwarzschild blackhole.

2.5 (d) Kerr-Newman blackhole:

In 1965, Ezra T. Newman obtained the most general solution of Einstein equation for charged and rotating body. The solution is given by

$$ds^2 = -\frac{\Delta_1}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta_1}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a\, dt]^2$$

$with \quad G = c = 1$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.18)$$

Where
$$\Delta_1 = r^2 - 2Mr + a^2 + Q^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$
$$a = \frac{J}{M}$$

The metric (2.18) is Known as Kerr-Newman metric or Kerr-Newman blackhole and it is a generalization of Kerr metric for uncharged rotating body which had been discovered by Roy Kerr two years ago.

The singularity of equ.(2.18) is given by as usual , $\Delta_1 = 0$ which gives

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2 - a^2}, \quad with \quad a^2 + Q^2 \le M^2 \ldots\ldots\ldots\ldots(2.19)$$

Equation (2.19) gives the equation of the event horizons of Kerr-Newman blackhole. It will be extremal when $a^2 + Q^2 = M^2$, while the Schwarzschild blackhole can never be extremal. To obtain this solution it is assumed that the cosmological constant equals to zero.

Table- 2.2 Properties of blackhole.

Name of the blackhole	Physical properties	Mathematical description
Schwarzschild blackhole	(i) no angular momentum (ii) no charge (iii) no energy extraction (iv) never extremal	(i) $J = 0$ (ii) $Q = 0$ (iii) $R_s = 2M$ (iv) $M^2 = M_{ir}^2$ (v) $A = 4\pi R_s^2 = 16\pi M^2$
Ressiner-Nordstrom blackhole	(i) no angular momentum (ii) charged (iii) energy can be extracted by reducing net charge. (iv) may extremal.	(i) $J = 0$ (ii) $Q \neq 0$ (iii) $r_+ = M + \sqrt{M^2 - Q^2}$ (iv) $A = 4\pi r_+^2 = 4\pi[M + \sqrt{M^2 - Q$ (v) $M^2 = M_{ir} + (\frac{Q^2}{4M_{ir}})^2$ (vi) $M = \|Q\|$
Kerr blackhole	(i) having angular momentum (ii) no charge (iii) energy can be extracted by reduction of angular momentum (iv) may extremal	(i) $J \neq 0$ (ii) $Q = 0$ (iii) $r_+ = M + \sqrt{M^2 - a^2}$ (iv) $A = 4\pi(r_+^2 + a^2)$ $= 4\pi[(M + \sqrt{M^2 - a^2})^2 + a^2]$ (v) $M^2 = M_{ir}^2 + \frac{J^2}{4M_{ir}^2}$ (vi) $\|a\| = M$
Kerr-Newman blackhole	(i) having angular momentum (ii) charged (iii) energy can be extracted by reduction of spin and charge (iv) may extremal	(i) $J \neq 0$ (ii) $Q \neq 0$ (iii) $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$ (iv) $A = 4\pi(r_+^2 + a^2)$ $= 4\pi[(M + \sqrt{M^2 - Q^2 - a^2})^2 + a$ (v) $M^2 = M_{ir} + (\frac{Q^2}{4M_{ir}})^2 + \frac{J^2}{4M_{ir}^2}$ (vi) $a^2 + Q^2 = M^2$

2.5 (e) BTZ blackhole:

In 1992, Maximo Banados, Claudio Teitelboim and Jorge Zanelli discovered the blackhole solution of Einstein equations in 2+1 dimension spacetime with a negative cosmological constant. This blackhole characterized by mass , angular momentum and charge ,define by flux integrals at infinity- is quite similar to its 3+1 dimensional blackhole counter part.

The BTZ blackhole metric is given by,

$$ds^2 = -N^2(r)\,dt^2 + N^{-2}(r)\,dr^2 + r^2[N^{\phi}(r)\,dt + d\phi]^2 \;\dots\dots\dots\dots\dots\dots (2.20)$$

with cosmological constant $\Lambda = -l^{-2}$.

Here the squared lapse $N^2(r)$ and the angular shift $N^{\phi}(r)$ are given by

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \quad , \quad N^{\phi}(r) = -\frac{J}{2r^2} \;\dots\dots\dots\dots\dots\dots (2.21)$$

Where $-\infty < t < \infty$, $0 < r < \infty$ and $0 \le \varphi \le 2\pi$.

The singularity of equation (2.20) is given by

$$N^2(r) = 0 \;\dots\dots\dots\dots\dots\dots (2.22)$$

Which gives

$$r_{\pm} = l\sqrt{\frac{M}{2}\left(1 \pm \sqrt{1-\left(\frac{J}{Ml}\right)^2}\right)} = l\sqrt{\frac{M}{2}(1 \pm \Delta)} \;\dots\dots\dots\dots\dots\dots (2.23)$$

Where $\Delta = \sqrt{1-\left(\frac{J}{Ml}\right)^2}$ with imposed the condition

$$M > 0 \quad and \quad |J| \le Ml \;\dots\dots\dots\dots\dots\dots (2.24)$$

Equation (2.23) gives the equation of event horizons of BTZ blackhole. We see that like Kerr blackhole , a rotating BTZ black hole contains an inner and an outer horizons.

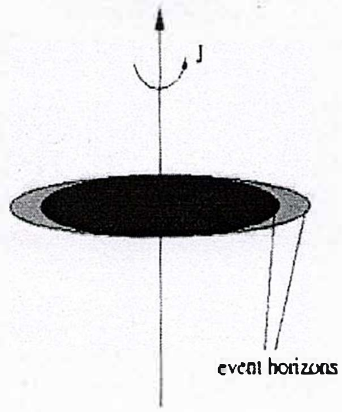

Figure: 2.9 The 2+1 dimensional BTZ blackhole. This blackhole can be visualized as a circular disc with spin J.

In the extremal case $|J| = Ml$, the roots of $N^2(r) = 0$ are coincide. The radius of the curvature $l = (-\Lambda)^{-\frac{1}{2}}$ provides the length scale necessary in order to have a horizon in which the mass is dimensionless. If one lets l grow very large the blackhole exterior is pushed away to infinity and one is left just with the inside[60].The vacuum space is obtained by setting $M \to 0$ which requires $J \to 0$ is

$$ds^2_{vac.} = -\frac{r^2}{l^2}dt^2 + (\frac{r^2}{l^2})^{-1}dr^2 + r^2 d\phi^2 \ldots\ldots\ldots\ldots\ldots\ldots(2.25)$$

The BTZ blackhole has an ergosphere and an upper bound in angular momentum for any given mass. The thermodynamic properties of BTZ blackholes is analogus to the 3+1 dimensional blackhole. For example, its entropy is captured by a law directly analogous to the Bekenstein bound in 3+1 dimensional, essentially with the surface area replaced by the BTZ blackholes circumfurrence.

If we set , $\Lambda = -1$ *i.e.* $l = 1$ for the spinless $(J = 0)$ BTZ blackholes we obtain from (2.20) as

$$ds^2 = -(-M + r^2)dt^2 + (-M + r^2)^{-1}dr^2 + r^2 d\phi^2 \ldots\ldots\ldots\ldots(2.26)$$

The metric (2.26) is singular at $r = \sqrt{M}$ .Thus $r = \sqrt{M}$ is an event horizon of the spinless BTZ blackhole.

2.6 Blackhole Theorems:

The natural outcome of the solutions of the Einstein's equation are blackholes and in theoretical physics, they have a fundamental importance. A number of important theorems on classical blackholes have been discovered in the last fifty years. The theorems are as follows:

(a) Singularity theorem (1965).
(b) Area theorem(1972).
(c) Uniqueness theorem (1975).
(d) Positive energy theorem (1983).
(e) Horizon mass theorem (2005).

2.6 (a) Singularity theorem:

This theorem developed by S. W. Hawking and R. Penrose[61,62] that quantify the specific conditions under which singularities are the inevitable result of the solutions of the Einstein's equation in general relativity. These theorems demonstrate that in the framework of general relativity, every blackhole must contain a singularity at its center and all expanding universe like ours must have begun with a big bang singularity.

In 1965, R. Penrose used the methods of global analysis to show that the singularities are general phenomena which occur in gravitational collapse irrespective of symmetry. The Penrose singularity theorems showed that a generic solution of Einstein's equations which satisfies certain reasonable physical conditions and contains a closed trapped surface is singular in the sense that it is geodesically incomplete. Thus although the theorems show that gravitational singularities are a general feature of gravitational collapse they do not say very much about the nature of the singularity. Although it is likely that the end point of a realistic collapse is a situation similar to the Schwarzschild singularity, in which there is a region where the gravitational force becomes unbounded and crush matter in infinite density, other sorts of weaker singularity are possible.

Hawking's singularity theorem is for the whole universe and works backwards-in-time. According to this theorem, our universe had its origins in a singularity. In the beginning all of the matter in the universe was concentrated in a single point, making a very small but tremendously dense body. This body exploded in a big bang that initiated time and the universe. Thus time has a beginning in the big bang and an end in a blackhole. The existence of a singularity shows that general relativity breaks down at the Planck scale as Hawking says, " The singularity theorems seem to imply that either the general theory of relativity breaks down or that there could be particles whose histories did not exist before a certain time. My own opinion is that the theory probably breaks down but only when quantum gravitational effects become important".

2.6 (b) Area theorem:

This theorem discovered by S.W.Hawking in 1972[63]. The theorem states that the total surface area of the outer event horizon of a blackhole can only stay the same or increase, but will never decrease in any classical process .This situation is exactly like the second law of thermodynamics which states that the entropy of a system can only stay the same or increase, but can never decrease. Eventually it is recognized that the area of a blackhole is its entropy and the surface gravity is its temperature.

2.6 (c) Uniqueness theorem:

The blackhole uniqueness theorem states that a blackhole is uniquely specified by its mass, charge and angular momentum[64]. The theorem is also known as the 'no hair' theorem. Thus there are only three types of blackholes; the neutral Schwarzschild blackhole, the charged Reissner-Nordström blackhole and the rotating Kerr blackhole. Two blackholes which have the same mass, charge and angular momentum are therefore indistinguishable to an external observer.

The blackhole uniqueness theorem first announced by Werner Israel at a meeting at Kings college London in 1967[65]. He had investigated an interesting class of static assymptotically flat solutions of Einstein's vacuum field equations. The solutions had a regular event horizons and obeyed the type of regularity conditions that a broad class of non-rotating equilibrium blackhole metrics might plausibly be expected to satisfy. His striking conclusion was that the class was exhausted by the positive mass Schwarzschild family of metrics. This result initiated research on the blackhole uniqueness theorems which continues today.

2.6 (d) Positive energy theorem:

In general relativity, the positive energy theorem states that an isolated gravitational system with non-negative local matter density must have non-negative total energy, measured at spatial infinity. In other words ,the mass of a blackhole is always positive[66]. Since gravity is an attractive force and the gravitational potential energy is always negative, so the question arises whether the gravitational binding energy of a blackhole is so great that it dominates over matter such that the total energy of the system becomes negative. The answer is that it cannot be.

2.6 (e) Horizon mass theorem:

The mass of a blackhole depends on where the observer is. The closer one gets to a blackhole the less gravitational energy one sees. As a result , the mass of a blackhole increases as one gets near the horizon. This is the latest theorem on blackholes called the horizon mass theorem[67]. The theorem states that for all the blackholes; neutral, charged or rotating, the horizon mass is always twice the irreducible mass observed at infinity i.e. $M(r_+) = 2M_{irr}$. The horizon mass $M(r_+)$ is the mass which cannot escape from the horizon of a neutral, charged or rotating blackhole. It is the mass observed at the horizon. The irreducible mass is the final mass of a charged or rotating blackhole when its charge or angular momentum is removed by adding external particles to the blackhole. It is the mass observed at infinity.

2.7 Observational evidence of blackhole:

In 1784, John Michell wrote in his famous article:

" If there should really exist in nature any (such) bodies,...we could have no information from sight ; yet,if any other luminuous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with same degree of probability,as this might afford a clue to some of the apparent irregularities of the revolving bodies,which could not be easily explicable on any other hypothesis."[68]

At the very beginning the theoretically predicted properties of blackhole were discussed but there were no observational evidence at that time. Following the Michell paper, the same argument stated by Laplace in 1798. There is a long gap until 1915 when with the coming of the General relativity theory by Einstein, the theoretical discussion of blackholes started a new.

Since the blackhole is itself invisible because nothing even light cannot escape from it, so it is very difficult to find a blackhole in nature. A blackhole can be found indirectly by observing its effect on the stars and gas close to it. After discovery of radio astronomy and X-ray astronomy, the observational search for blackholes get a new dimension. At present, observational evidence supports the idea that blackholes occur ubiquitously in nature. Two kinds of blackholes are observed: stellar-sized blackholes in x-ray binary systems, mostly in our own Milky Way galaxy, and supermassive blackholes in Active Galactic Nuclei(AGN)found at the centers of our won and other galaxies.

One of the most important evidence for existence of blackhole is binary system of stars. In this system the stars are very close to each other. If one of this stars explodes catastrophically as a supernova and forms a blackhole, then the gas and dust of the other star might be pulled towards the star which form blackhole. This gas and dust begin to orbit around the event horizon and then orbit the blackhole. The gas becomes heavily compressed and the frictions among the atoms converts the kinetic energy of the gas and dust into heat and X-rays are emitted. From this orbiting material radiations scientists can measure its heat and speed. From the motion and speed of the circulating matter, scientists can infer the presence of a blackhole .

During the 1970s and 1980s , particular attention was focussed on the source Cygnus X-1, which appeared to be the strongest candidate for containing a blackhole. From this source, Cygnus X-1/HDE226868 identified as a binary blackhole system, orbiting an unseen companion with an orbital period of 5.6 days. It showed X-ray variability on a range of timescales extending down to one millisecond , indicating that the companion is extremly compact and must be a neutron star or a blackhole.But neutron stars cannot have arbitrarily large mass; there is a maximum above which the pressure can no longer balance gravity. This maximum mass lies between $1.4M_{\otimes}$ and $2.5M_{\otimes}$ if the neutron star is non-rotating and may be raised by up to 25% if it is rotating rapidly. If one could determine that the mass of a very compact object is above the maximum for a neutron star, then it would presumably have to be a blackhole. This is the line of reasoning that was follwed with Cygnus X-1 and with various subsequent blackhole candidates[69].

We know that HDE 226868 is a member of a binary system because its spectrum shows systematic Doppler shifts which can consistent with it moving on a binary orbit under the influence of the unseen companion. From the Doppler shift data, a radial velocity curve can be constructed, giving the variation with time of the component of the star's velocity along the line of sight. From this one can extract the orbital period P, the radial velocity amplitude K and, in principle, the eccentricity of the orbit . Kepler's laws gives the following mass function which relates observed quantities to unknown masses:

$$f(M_X) = \frac{PK^3}{2\pi G} = \frac{M_X^3 \sin^3 i}{(M_X + M_c)^2} \quad \text{.................................(2.27)}$$

where M_X is the mass of the compact object, M_c is the mass of the companion star and i is the inclination angle. A crucial fact is that M_X cannot be less than the value of the mass function. Therefore the best blackhole candidates are obtained when the observed mass function exceeds $3M_\otimes$ - since, according to the theory, a neutron star more massive than this limit is unstable and will collapse to form a blackhole. Otherwise , additional information is necessary to deduce M_X: the spectral type of the primary gives approximately M_c, the presence or absence of X-ray eclipses gives bounds to $\sin i$. Hence M_X is obtained within some error bar. Blackhole candidates are retained only when the lower limit exceeds $3M_\otimes$. At present day, about ten binary X-ray sources provide good blackhole candidates. They can be divided into two families : the high mass X-ray binaries(HMXB), where the companion star is of high mass, and the low mass X-ray binaries(LMXB) where the companion is typically below a solar mass. The latter are also called "X-ray transients" because they flare up to high luminosities[70].

In 1989, the X-ray satellite Ginga discovered a new XRT(X-ray transient)in outburst named GS2023+338/V 404 Cygni, identify the binary blackhole candidate. Many hundreds of X-ray binary systems are known in our Milky Way galaxy, but only 10s of these have measured masses, and in about 20 the measured mass indicates a blackhole. Table 2.3 presents the current list of 20 confirmed blackholes based on dynamical arguments,ordered by orbital peroid[71].

Table- 2.3 (Confirmed blackholes and mass distributions)

System	P_{orb} (days)	f(M) $[M_{\otimes}]$	Donor Spect. Type	Classification	M_X $[M_{\otimes}]$
GRS 1915+105	33.5	9.5 ± 3.0	K/M III	LMXB/Transient	14 ± 4
V404 Cyg	6.471	6.09 ± 0.04	KO IV	,,	12 ± 2
Cyg X-1	5.600	0.244 ± 0.005	09.7 Iab	HMXB/Persistent	10 ± 3
LMC X-1	4.229	0.14 ± 0.05	07 III	,,	> 4
XTE J1819-254	2.816	3.13 ± 0.13	B9 III	IMXB/Transient	7.1 ± 0.3
GRO J1655-40	2.620	2.73 ± 0.09	F3/5 IV	,,	6.3 ± 0.3
BW Cir	2.545	5.74 ± 0.29	G5 IV	LMXB/Transient	> 7.8
GX 339-4	1.754	5.8 ± 0.5	-	,,	-
LMC X-3	1.704	2.3 ± 0.3	B3 V	HMXB/Persistent	7.6 ± 01.3
XTE J1550-564	1.542	6.86 ± 0.71	G8/K8 IV	LMXB/Transient	9.6 ± 1.2
4U 1543-475	1.125	0.25 ± 0.01	A2 V	IMXB/Transient	9.4 ± 1.0
H1705-250	0.520	4.86 ± 0.13	K3/7 V	LMXB/Transient	6 ± 2
GS 1124-684	0.433	3.01 ± 0.15	K3/5 V	,,	7.0 ± 0.6
XTE J1859+226	0.382	7.4 ± 1.1	-	,,	-
GS2000+250	0.345	5.01 ± 0.12	K3/7 V	,,	7.5 ± 0.3
A0620-003	0.325	2.72 ± 0.06	K4 V	,,	11 ± 2
XTE J1650-500	0.321	2.73 ± 0.56	K4 V	,,	-
GRS 1009-45	0.283	3.17 ± 0.12	K7/M0 V	,,	5.2 ± 0.6
GRO J0422+32	0.212	1.19 ± 0.02	M2 V	,,	4 ± 1
XTE J1118+480	0.171	6.3 ± 0.2	K5/M0 V	,,	6.8 ± 0.4

In 1992, a supermassive blackhole detected by Hubble Space Telescope (HST) at the heart of the active galaxy M87 located 50 million light years from earth in the constellation Virgo. Also an instrument aboard the Hubble Space Telescope called the Space Telescope Imaging Spectograph(STIS) was installed in February 1997. It is mainly ‘blackhole hunter’. STIS found the signature of a supermassive blackhole in the center of the galaxy M84. The spectra showed a rotation velocity of 400km/s while earth orbits our sun at 30km/s. If the earth moves at fast as 400km/s then our year would be only 27 days long!

CHAPTER THREE
THERMODYNAMICS AND BLACKHOLE MECHANICS

3.1 Thermodynamics and Gravity:

Thermodynamics is a branch of physics which deals with the energy, heat, work and entropy of a system. It was born in 19th century as scientists were first discovering how to build and operate steam engines. Thermodynamics deals only with the large scale respond of a system which we can observe and measure in experiments. It is closely related to statistical mechanics from which many thermodynamic relationships can be derived. While dealing with process in which systems exchange matter or energy ,classical thermodynamics is not concerned with the rate at which such processes take place, termed kinetics. For this reason ,the use of the term 'thermodynamics' usually refers to equilibrium thermodynamics. In this connection a central concept in thermodynamics is that of ' quasistatic processes' which are idealized' infinitely slow' processes. Time dependent thermodynamic processes are studied by non-equilibrium thermodynamics.

The ordinary laws of thermodynamics are of very general validity and they do not depend upon the details of the underlying 'microscopic dynamics' of particular systems. This mean that they can be applied to systems about which one knows nothing other than the balance of energy and matter transfer between them and the environment. Example of this include Einstein's prediction of spontaneous emission around the term of the 20th century and the current research into the thermodynamics of black holes.

On the other hand, gravitation or gravity is a natural phenomenon in which objects with mass attract one another. Gravitation is most familiar as the agent that gives weight to objects with mass and causes them to fall to the ground when dropped. It is one of the four fundamental force of nature, along with the nuclear force or strong force, electromagnetic force and weak force. Einstein describes gravitations using the general theory of relativity, in which gravitation is a spacetime curvature instead of a force. He proposed that spacetime is curved by matter, and that free falling objects are moving along locally straight paths in curved spacetime.

From the above discussion it is clear that the topics of thermodynamics and gravity lead a rather separate existence in physics. In the broadest sense, thermodynamics regulates the organization of activity in the universe, and gravity controls the dynamics, at least on the large scale. The interaction between these conceptually dissimilar aspects of fundamental physics is still now full of paradoxes, muddle and uncharted hazards. The main difficulties about the thermodynamics of gravitating systems is the apparent absence of true equilibrium. Stars are hot, self-gravitating balls of gas inside which the weight of the star is supported by its won internal kinetic or zero-point quantum pressure. A star is made hotter, not by adding energy, but by removing it, which is unlike ordinary thermodynamic systems.

3.2 Laws of ordinary thermodynamics:

(a) Zeroth law:

The zeroth law of thermodynamics states that,

'if two systems in thermal equilibrium with a third system, then they are in thermal equilibrium to each other.'

This zeroth law is sort of a transitive property of thermal equilibrium. The transitive property of mathematics is if $A = B$ and $B = C$ then $A = C$. The same is true of thermodynamic systems that are in thermal equilibrium. Systems are said to be in equilibrium if the small, random exchanges between them do not lead to a net change in energy. At the beginning of the 20^{th} century, British physicist Ralph H. Fowler coined the term 'zeroth law' and this law is more fundamental even than the other laws.

(b) First law:

The first law of thermodynamics is an expression of the principle of the conservation of energy. It states that,

" the change in a system's internal energy is equal to the difference between heat added to the system from its surroundings and work done by the system on its surroundings."

This law gives a very simple idea. If heat is added to a system, then there are only two things that can be done – change the internal energy of the system or cause the system to do work. All of the heat energy must go into doing these things. Mathematical form of this law is

$$dU = TdS - PdV \qquad (3.1)$$

where U is the internal energy, T is the temperature, P is the pressure, S is the entropy and V is the volume.

(c) Second law:

The second law of thermodynamics is an expression of the universal principle of decay observable in nature. It states that,
" Heat cannot spontaneously flow from a colder location to a hotter location."
This law is formulated in many ways, as will be addressed shortly, but is basically a law which, unlike most other laws in physics deals not with how to do something, but rather deals with entirely with placing a restriction on what can be done.

In practical applications, this law means that any heat engine or similar device based upon the principles of thermodynamics cannot, even theoretically be 100% efficient. In 1824,French physicist and engineer Sadi Carnot discovered this principle, when he developed his Carnot Cycle engine and later German physicist Rudolf Clausius formalized it as a law of thermodynamics. This law is perhaps the most popular outside of the realm of physics, because it is closely related to the concept of entropy or the disorder created during the thermodynamic process. This law can be reformulated as a statement regarding entropy as reads,
"In any closed system, the entropy of the system will either remain constant or increase."
This is one definition used for the arrow of time, since entropy of the universe will always increase over time according to the second law of thermodynamics.

(d) Third law:

The third law of thermodynamics is essentially a statement about the ability to create an absolute temperature scale, for which absolute zero is the point at which the internal energy of a solid is precisely zero. This law states that,
"It is impossible to reduce any system to absolute zero in a finite series of operations."
Another statement of this law is,
" As a system approaches absolute zero, all processes cease and the entropy of the system approaches a minimum value."
The third law of thermodynamics is a statistical law of nature regarding entropy and the impossibility of reaching absolute zero of temperature. This law also provides an absolute reference point for the determination of entropy.

3.3 Blackhole and thermodynamics :

At the biginning of his famous book "The Mathematical Theory of Black Holes (1983)", Chandrasekhar remarking ,

> "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well."

Blackholes are perhaps 'the most perfect objects in the universe', because they are completely characterized by a small number of macroscopic parameters mass, charge and angular momentum. All the details of the matter that formed a blackhole becomes irrelevant as that matter passes through the event horizon i.e. the boundary of the blackhole ; there is no physical difference between any blackholes of equivalent mass, charge and angular momentum, regardless of countless ways such a blackhole can be formed.

Over the last forty years, blackholes have been shown to have a number of surprising properties. This properties have revealed unforeseen relations between the otherwise distinct areas of general relativity, quantum mechanics and statistical mechanics.This interplay, in turn, led to a number of deep puzzles at the very foundations of physics. Some have been resolved while others continue still now. The thermal properties of blackholes come from the behavior of their macroscopic properties that were formalized in the four laws of black hole mechanics by Bardeen, Carter and Hawking [1]. They dictate the behavior of blackholes in equilibrium , under small perturbations away from equilibrium , and in fully dynamical situations. Although, these laws are consequences of classical general relativity alone, but they have a close similarity with the laws of ordinary thermodynamics.The origin of this seemingly strange coincidence lies in quantum physics. Although this parallel was extremely suggestive, taking it seriously would require one to assign a non-zero temperature to a blackhole, while all agreed was absurd because blackhole by its very definition do not emit anything, so the only temperature one might be able to assign them is absolute zero. But this idea was overthrown by the discovery of Hawking radiation. He proposed that blackholes are not completely black and their physical temperature are not absolute zero[9].The surface gravity of blackholes can indeed be interpreted as a physical temperature.

At first in 1971, Hawking stated that the area , A of the event horizon of a blackhole can never decrease(but can remain constant) in any process;

$\Delta A \geq 0$ ....(3.2)

When radiation or matter falls through it, or when two blackholes coalesce, there is an increase in the total horizon area. In this respect it is much like the thermodynamic concept, entropy. The entropy of the universe can increase, but it can never decrease. It was later noted by Bekenstein [3] that this result is analogous to the statement of the ordinary second law of thermodynamics, namely that the total entropy , S of a closed system never decrease in any process;

$\Delta S \geq 0$ ....(3.3)

The above comparison suggests that it might be useful to consider blackhole physics from a thermodynamic view point; something like entropy may also play a role in it. The difference of these two laws are ; in thermodynamics one can transfer entropy from one system to another and it is required only that the total entropy does not decrease whereas in the case of blackhole, one cannot transfer area from one blackhole to another since blackholes cannot bifurcate. So the second law of black hole mechanics requires that the area of each individual blackhole does not decrease in any process. In this sense the second law of blackhole mechanics is slightly stronger than the corresponding thermodynamic law.

Bekenstein realized that considerable information was lost within the event horizon when the blackhole was formed. He suggested that the entropy of the blackhole could be related to the logarithm of this information. This information is , in fact , related to the surface area and it was eventually shown that the entropy of a black hole S_{bh} could be written as ;

$S_{bh} = \frac{c}{4\hbar} A k_B$ ....(3.4)

where A is the surface area of the event horizon, $\hbar$ is the Planck-Dirac constant $(\frac{h}{2\pi})$, k_B is Boltzman's constants.

Using this definition Bekenstein proposed the generalized second of thermodynamics to include blackholes as

$\Delta S_{bh} + \Delta S_c \geq 0$ ....(3.5)

Where S_c is the common entropy in the blackhole exterior.

It is already mentioned that Hawking discovered that the surface of a blackhole could not have a temperature of absolute zero. Mathematically it appeared to have a non-zero temperature. Hawking discovered by applying quantum mechanics to the region near the event horizon, that blackholes can emit all species of particles and radiation[72].

In particular , the spectrum of emission is given by;[73]

$$<n> = \frac{\Gamma}{e^{\frac{\hbar\omega}{kT}} - 1} \qquad (3.6)$$

Where $<n>$ is the mean number of quanta emitted in one mode of frequency ω, and Γ is the blackholes absorbivity. The surface temperature of black hole is given by;

$$T = \frac{\hbar\kappa}{2\pi ck_B} \qquad (3.7)$$

where κ is the surface gravity of the blackhole evaluated on the event horizon.[72]

After established that the blackholes have a non-zero surface temperature and an entropy it is easy to show that they also obey the zeroth, first and second laws of thermodynamics. It is also believed that they may also obey the third law in most but not necessarily all cases. To obey it in all cases requires that the 'cosmic cencorship hypothesis' be satisfied. This hypothesis was made by the British mathematician Roger Penrose, which states that when a star collapses to a singularity, this singularity is always concealed from the outside world by an event horizon. So far this has not been proved and still now it is the number one question in classical general relativity.

3.3(a) zeroth law of black hole mechanics:

This law states that " The surface gravity , κ of a stationary black hole is constant over the event horizon".

Although κ is defined locally on the event horizon, it turns out that it is always constant over the horizon of a stationary blackhole. This constancy is reminiscent of the zeroth law of thermodynamics which states that the temperature is constant throughout a body in thermal equilibrium. It suggests that the surface gravity is analogus to the temperature. T constant for thermal equilibrium for a normal system is analogous to κ constant over the horizon of a stationary blackhole. The surface gravity is related to the physical temperature of the blackhole , namely Hawking temperature is given by; [72]

$$T_H = \frac{\hbar\kappa}{2\pi k_B} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.8)$$

For the case of Schwarzschild black hole, where $\kappa = \frac{1}{4GM}$,the Hawking temperature becomes;

$$T_H = \frac{\hbar}{8\pi G k_B M} \approx 6.2\times10^{-8}(\frac{M_\otimes}{M})^0 K \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.9)$$

So this is completely negligible for solar mass black hole- the black hole absorbs much more from the microwave background radiation than it radiates itself. In the case of rotating Kerr black hole, the Hawking temperature is given by;

$$T_H = \frac{\hbar\kappa}{2\pi k_B} = 2(1+\frac{M}{\sqrt{M^2-a^2}})^{-1}\frac{\hbar}{8\pi M k_B} < \frac{\hbar}{8\pi M k_B} \quad \ldots\ldots\ldots\ldots(3.10)$$

where a and κ are defined by ; [8]

$a = \frac{J}{M}$ and $\kappa = \frac{2\pi(r_+ - r_-)}{A}$. Similarly one can obtain Hawking temperature for other blackholes.

We see that the Hawking radiation plays no roles in the case of large-sized black holes. The only type of black hole where one can hope to observe such the radiation is the so called 'mini black hole' which are created along with the universe in the early stage.

3.3 (b) First law of black hole mechanics:

This law deals with the mass change, dM when a black hole undergo from one stationary state to another. Mathematical formulation of this law is given by;

$$dM = \frac{\kappa}{8\pi}dA + work\ term \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.11)$$

Or,

$$dM = T_H dS_{bh} + work\,term \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.12)$$

The entropy of the black hole is thus represented by a quarter of the area of the event horizon, that is , $S_{bh} = \frac{A}{4}$ ……………………………(3.13)

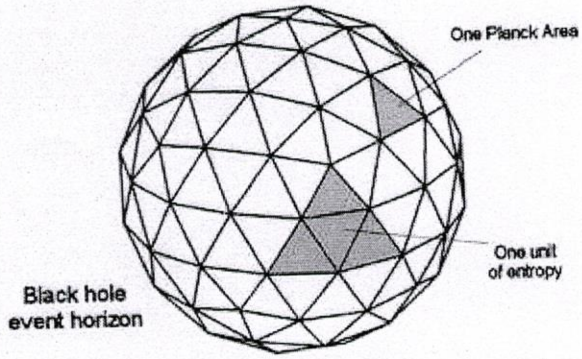

Figure:3.1 The Bekenstein-Hawking entropy is the entropy to be ascribed to any black hole: one quarter of its horizon area expressed in units of the Planck area.[From, scholarpedia]

The factor $\frac{1}{4}$ was indeed found by Hawking, when he applied quantum field theory to the black holes which shows that they will absorbs and emit particles as if they were thermal bodies with the Hawking temperature given by equation (3.8).

The 'work terms' depends on the type of blackholes. For the most general type Kerr-Newman black hole family, the first law takes the form;

$$dM = \frac{\kappa}{8\pi} dA + \Omega\, dJ + \Phi\, dQ \qquad (3.14)$$

where Ω is the angular velocity and Φ is the electric potential which are given by

$$\Omega = \frac{\delta M}{\delta J} \qquad (3.15)$$

and

$$\Phi = \frac{\delta M}{\delta Q} \qquad (3.16)$$

3.3 (c) Second law of black hole mechanics:

The second law of black hole mechanics is Hawking area theorem[8].This law states that, in any classical process the area of the event horizon does not decrease with time i. e.

$$dA \geq 0 \quad or \quad dS_{bh} \geq 0 \qquad (3.17)$$

This law implies for instance that the area of a blackhole resulting from the coalescence of two parent blackholes is greater than the sum of areas of the two parent blackholes. It also implies that the blackholes cannot bifurcate, namely a single blackhole can never split in two parts.

The magnitude of the blackhole entropy is very large. In fact , the blackhole state is the maximum entropy state of a given amount of matter. If we express dimension fully, the Bekenstein-Hawking entropy of the blackhole is given by

$$S_{BH} = \frac{Ak_B}{4G\hbar} \qquad (3.18)$$

For Schwarzschild blackhole, this gives

$$S_{BH} = \frac{k_B \pi R_s^2}{G\hbar} \qquad (3.19)$$

Numerically the entropy of the sun is $S_{\otimes} \approx 10^{57} k_B$, whereas a solar mass black hole has an entropy of about $10^{77} k_B$ which is 20 orders of magnitude larger!

3.3 (d)Third law of black hole mechanics:

The third law of blackhole mechanics states that, 'it is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.' This law has a rather different status from the others, in that it does not, so far at least, have a rigorous mathematical proof.[1] However ,for example if one tries to reduce κ of Kerr black hole by throwing in particles to increase the angular momentum, one finds that the decrease of κ per particle thrown in gets smaller and smaller as the mass and angular momentum tend to the critical ratio $\frac{J}{M^2} = 1$ i.e. extremal case for which is κ zero. Actually $\kappa = 0$ is merely an idealized case because it is forbidden by the 'cosmic censorship hypothesis'.

3.4 Generalized second law (GSL):

The correspondence between the laws of ordinary thermodynamics and the laws of blackhole mechanics was treated as a mathematical curiosity without any physical implications, in a seminal paper by Bardeen, Carter and Hawking[1]. At around the same time , Bekenstein[3] was advocating a rather more radical approach. On the basis of blackhole's area theorem he proposed that , multiplied by appropriate powers of the Planck length, Boltzmann constant and some dimensionless constant of order unity, the blackhole area should be interpreted as its physical entropy. This proposal was given physical support by the discovery of Hawking[9] that the blackholes radiate at a temperature $T_H = \frac{\hbar\kappa}{2\pi}$.

Wheeler provided the initial motivation for Bekenstein's blackhole entropy proposal[76]. Wheeler suggested a creature, subsequently called Wheeler's demon, which could violate the ordinary second law of thermodynamics by dropping entropy into a blackhole, producing a decrease in the entropy outside the blackhole. This led Bekenstein to conjecture that the blackhole itself has an entropy.

Wald[2] gives an explanation which further strengthed the physical connection between the laws of blackhole mechanics and the laws of thermodynamics by the following considerations. If we take into account the 'back reaction' of the quantum field on the blackhole, then it is clear that if energy is conserved in the full theory, an isolated blackhole must lose mass in order to compensate for the energy radiated to infinity in the particle creation process. As a blackhole thereby "evaporates", the blackhole entropy S_{bh} will decrease, in violation of the second law of blackhole mechanics. On the other hand , there is a serious difficulty with the ordinary second law of thermodynamics when blackholes are presents: one can simply take some ordinary matter and drop it into a blackhole, where, classically at least, it will disappear into a spacetime singularity. In this later process, one loses the entropy initially present in the matter, but no compensating gain of ordinary entropy occurs, so the total entropy , S , decreases. It is notable that in the blackhole evaporation process, although S_{bh} decreases, there is significant amount of ordinary entropy generated outside the blackhole due to particle creation. Similarly , when ordinary matter is dropped into a blackhole, although S decreases, by the first law of blackhole mechanics , there will necessarily be an increase in S_{bh}.

The above considerations motivated Bekenstein to take the following proposal[3,77]. Although the second law of blackhole mechanics breaksdown when quantum process are considered, and the ordinary second law of thermodynamics breaksdown when blackholes are present, perhaps the following law, known as the generalized second law (GSL) always holds. The law stated that,

" In any process, the total generalized entropy never decreases"
This statement means that we must regard blackhole entropy as a genuine contribution to the entropy content of the universe. [3]
If we define the total generalized entropy by S' then

$$S' = S_{bh} + S_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.20)$$

where S_{bh} is the blackhole entropy and S_c is the common entropy in the blackhole exterior and then GSL becomes

$$\Delta S' = \Delta(S_{bh} + S_c) \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.21)$$

Although S_{bh} and S_c individually may decrease, it appears to be true that S' never decreases. If we decrease S_c by throwing matter into a blackhole, we correspondingly increase A i.e. S_{bh} so that S' does not decrease. On the other hand, if A i.e. S_{bh} decreases due to the quantum particle creation processes then the thermal spectrum of the created particles increase S_c; again S' does not decrease. Thus neither the second law of thermodynamics nor the blackhole area theorem are satisfied individually, but it appears that we have a new law of physics namely GSL.

The generalized entropy (3.20) and the generalized second law (3.21) have obvious interpretations: Presumably, for a system containing a blackhole, S' is nothing more than the " true total entropy" of the complete system , and (3.21) is then nothing more than the "ordinary second law" for this system. If so, then S_{bh} truly is the physical entropy of a blackhole.

3.4.1 Validity of GSL:

The GSL plays a fundamental role in blackhole physics. Though a number of analysis [78,79,80,81] have given strong support to the GSL but a simple explicit general proof of this law has not been given until now. Although these analysis have been carried out in the context of general relativity, the arguments for the validity of the GSL should be applicable to a general theory of gravity, provided, of course, that the second law of blackhole mechanics holds in classical theory.

The validity of the GSL for the massless radiation evaporated by an uncharged, non-rotating semi-classical blackhole was almost proved by Zurek[81]. Unruh and Wald [78] stressed the importance of the vaccum polarization and acceleration radiation effects for the validity of the GSL . More general arguments for the validity of this law for slowly evolving blackholes were given by Zurek and Thorne[79]. Also a simple explicit proof of the GSL for quasistationary changes of a generic charged rotating blackhole emitting, absorbing, and scattering any sort of radiation in the Hawking semi-classical formalism were given by Frolov and Page[80]. They assumed that the incoming state is a product state of radiation originating from infinity(i.e. IN modes) and radiation that would appear to emanate from the whitehole region of the analytically continued spacetime(i.e. UP modes), and it is argued that the generalized entropy must increase under unitary evolution. This is an explicit mathematical demonstration of what Zurek, Thorne and Price [85] argued verbally, that the GSL is a special case of the ordinary second law, with the blackhole as a hot, rotating, charged body that emits thermal radiation uncorrelated with what is incident upon it. Sorkin [86] argued on quite general grounds that the (generalized) entropy of the state of the region exterior to the blackhole must increase under the assumption that it undergoes autonomous evolution.

Most of the proofs of the GSL based upon two key assumptions[84]. One of the assumption is that the blackholes might be quasistationary, changing only slowly during its interaction with an environment. It has been conjectured [85] that the GSL also holds , using the Bekenstein-Hawking entropy formula $\frac{A}{4}$ for the blackhole, even for rapid changes in the blackhole, but this has not been rigorously proved. Another assumption is that the semiclassical approximation holds, so that the blackhole described by a classical metric which responds only to some average or expectation value of the quantum stress-energy tensor. This allows the blackhole entropy to be represented by $\frac{A}{4}$ of its classical horizon. This approximation also implies that the radiation from the blackhole is essentially thermal, with negligible correlations between what is emitted early and late in the radiation, so that one may use the von Newmann entropy $S_{rad} = -tr(\rho \ln \rho)$ for the entropy of the radiation and yet have it plus $\frac{A}{4}$ for the blackhole to continue to increase[84].

It is notable that if one could violate the GSL for an infinitesimal quasi-static process in a regime where the blackhole can be treated semi-classically, then it also be possible to violate the ordinary second law for a corresponding process involving a self-gravitating body. For example, suppose that the GSL could be violated for an infinitesimal quasi-static process involving, say, a Schwarzschild blackhole of mass M(with M much larger than the Planck mass). This process might involve lowering matter towards the blackhole and possibly dropping the matter into it. However , an observer doing this lowering or dropping can examine only the region outside of the blackhole, so there will be some $r_0 > 2M$ such that the detailed structure of the blackhole will directly enter the analysis of the process only for $r > r_0$. Now replace the blackhole by a shell of matter of mass M and radius r_0 and surround this shell with a "real" atmosphere of radiation in thermal equilibrium at the Hawking temperature as measured by an observer at infinity. Then the ordinary second law should be violated when one performs the same process to the shell surrounded by the "real" thermal atmosphere as one performs to violate the GSL when the blackhole is present. Indeed , the arguments of [79,85,87] do not distinguish between infinitesimal quasi-static process involving a blackhole as compared with a shell surrounded by a thermal atmosphere at the Hawking temperature. Wald [82] conclude that there appear to be strong ground for believing in the validity of the GSL.

3.4.2 Problems of GSL:

Before Hawking's discovery that the blackholes radiate, Bekenstein realized [3,77]that there is a serious difficulty with the GSL. One considers a process wherein one carefully lowers a box containing matter with entropy S and energy E very close to the horizon of a blackhole before dropping it in. Classically , if one could lower the box arbitrarily close to the horizon before dropping it in , one would recover all of the energy originally in the box as "work" at infinity. No energy would be delivered to the blackhole , so by the first law of blackhole mechanics, equation (3.14) , the blackhole area A , would not increase. However , one would still get rid of all of the entropy, S,originally in the box, in violation of the GSL[82].

To avoid this violation of the GSL, Bekenstein proposed that there was a limit on how close to the blackhole an object with fixed entropy S and fixed local energy E could be lowered. This led Bekenstein[83] to conjecture that the entropy S of a system of energy E and linear size R was limited by the formula,

$S \leq 2\pi E R$ ....(3.22)

The above proposal is known as Bekenstein's "entropy bound". Though this proposal recovery the difficulties of GSL but it developed that there are a lot of problem with it. The main difficulty is how to give precise definitions for the system and for its S, E, and R [84].

An alternative resolution was proposed in [78], based upon the idea that, when quantum effects are taken into account, the physical temperature of a blckhole is no longer absolute zero, but rather is the Hawking temperature, $T_H = \frac{\hbar\kappa}{2\pi}$. Since for a large blackhole, Hawking temperature goes to zero so it might appear that quantum effects could not be of much relevance in this case. However , despite the fact that Hawking radiation at infinity is indeed negligible for large blackholes, the effects of the quantum "thermal atmosphere" surrounding the blackhole are not negligible on bodies that are quasi-statically lowered toward the blackhole. The temperature gradient in the thermal atmosphere implies that there is a pressure gradient and , consequently, a buoyancy force on the box. This buoyancy force becomes infinitely large in the limit as the box is lowered to the horizon. As a result of this buoyancy force, the optimal place to drop the box into the blackhole is no longer the horizon but rather the 'floating point' of the box , where its weight is equal to the weight of the displaced thermal atmosphere. The minimum area increase given to the blackhole in the process is no longer zero, but rather turns out to be an amount just sufficient to prevent any violation of the GSL from occurring in this process[78].

3.5 Analogy between blackhole mechanics and thermodynamics:

Mathematically, the laws of blackhole mechanics completely analogous to the laws of ordinary thermodynamics. Although the nature of the laws of blackhole mechanics is completely different from the nature of the laws of thermodynamics, so it is generally believed that the analogy between them is purely a mathematical curiosity. But the discovery of particle creation by blackholes and their evaporation suggest that there may be a deep connection between blackhole mechanics and thermodynamics.

The analogy with thermodynamic behavior is striking, with the horizon area playing the role of entropy. This analogy was vigorously pursued as soon as it was recognized at the beginning of the 1970's.However, the caution should be used in developing the analogy, it appeared at first some flaws such as;

(i) the temperature of a blackhole vanishes.
(ii) the entropy is dimensionless, whereas horizon area is a length squared.

Table-3.1
Analogy between thermodynamic parameter and black hole's parameter.

Thermodynamic system	Black hole mechanics
Temperature, T	Surface gravity, κ
Energy , U	Black hole mass, M
Entropy , S	Area of the event horizon, A

(iii) the area of every blackhole is separately non-decreasing, whereas only the total entropy is non-decreasing in thermodynamics.

(iv) the GSL can be violated by adding entropy to a blackhole without changing its area.

At the purely classical level, it thus appear that the GSL is simply not true. However , when $\hbar \to 0$, the Bekenstein entropy $\frac{\eta A}{\hbar G}$ diverges , and an infinitesimal area change can make a finite change in the Bekenstein entropy. The other flaws [(i),(ii),(iii)] in the thermodynamic analogy are also in a sense resolved in the limit $\hbar \to 0$. The second flaw is resolved by the Bekenstein's postulate(multiplied by appropriate powers of the Planck length, Boltzmann constant and some dimensionless constant of order unity, the blackhole area should be interpreted as its physical entropy) , while third flaw is resolved because a finite decrease in area would imply an infinite decrease in entropy. Furthermore , the first law of blackhole mechanics implies that the blackhole has a Bekenstein temperature $T_B = \frac{\hbar \kappa}{8\pi\eta}$, which vanishes in the classical limit when $\hbar \to 0$, thus resolving first flaw. The Bekenstein proposal therfore "explains" the apparent flaws in the thermodynamic analogy, and it suggests very strongly that the analogy is much more than an analogy. It turns out that , with quantum effects included, the GSL is indeed true after all,with the coefficient η equal to $\frac{1}{4}$ [88].

The most obvious analogy between blackhole mechanics and thermodynamics is the second law. This law states that the area, A of the event horizon around a blackhole never decrease with time. When two blackholes coalesce, the area of the event horizon around the final blackhole is greater than the sum of the areas of the horizons of the original blackholes, i.e. $A_3 > A_1 + A_2$.

This law shows that the area of the event horizons has a strong similarity to entropy because it is additive and non-decreasing. It is mentioned above that the only difference between horizon area and entropy is that, one can transfer entropy from one system to another but in the case of blackhole one cannot transfer area from one blackhole to another because blackholes can never divide into two, they only joined together.[74]

Consider, the most general case of blackholes i.e. Kerr-Newman blackholes that characterized by mass M, angular momentum J and electric charge Q, the size of the blackhole area A is given by,

$$A = 4\pi(r_+^2 + a^2) = 4\pi[2M^2 - Q^2 + 2M^2(1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4})^{\frac{1}{2}}] \qquad (3.22)$$

With $Q^2 < M^2$, $J^2 < M^4$, $G = c = 1$

We see from equation (3.22) that, it is not clear at a glance whether a disturbance to the blackhole which changes both Q and J as well as mass M, will always increase the total area A. Consider the Penrose energy extraction process from a rotating and charged blackhole by reducing both Q and J. The mechanism of this process is of propelling a small body into the region just out side the event horizon where some particle trajectories possess negative energy relative to infinity .When the body reaches the ergosphere arrange for it to break apart into two fragments in such a way that one of which has negative energy and this part disappears down the hole. As a result it will reduce the total mass M of the blackhole somewhat and the mass-energy thereby released by this sacrificed components appears in the remaining fragment which is ejected to infinity at high speed. During this energy transfer the blackhole's rotation rate is diminished somewhat, so J also decreases. The equation (3.22) shows that when J decreases, the area A increases but when M decreases, the area A decreases. The changes in M and J are therefore in competition, but a careful calculation shows that J always wins and the area increases[75]. So there is a strong analogy between event horizon area and entropy i.e. the second law of blackhole mechanics and thermodynamics

Table-3.2 Analogy between the laws of thermodynamics and the laws of blackhole mechanics.

Law	Thermodynamics	Black hole mechanics
Zeroth	Equality of temperature is a condition for thermodynamic equilibrium between two systems or between two parts of the same system.	For any blackhole, κ is consta over the event horizon.
First	In an isolated system,the total energy of that system is conserved.	In an isolated system includir blackholes,the total energy of system is conserved.
Second	During any process the entropy of an isolated system increases or remain the same.	The surface area of a blackhc always remains constant or increases during any process.
Third	It is impossible to reduce the temperature of a system to zero by a finite number of processes.	It is impossible to reduce κ o blackhole to zero by a finite number of processes.

Now from equation (3.22) one can obtain;

$$dM = \frac{\kappa}{8\pi} dA + \Omega\, dJ + \Phi\, dQ \quad \text{.................................(3.23)}$$

Where $\frac{\kappa}{8\pi} \equiv \frac{\delta M}{\delta A}$ etc. The equation (3.23) just an expression of mass-energy conservation and corresponding to the first law of thermodynamics. In equation (3.1) the term PdV represents the work term whereas in equation (3.23) the term ΩdJ represents the work done on the spin and the term ΦdQ represents the work done on the electric field. So we can re-write the first law of thermodynamics as,

$$dU = TdS + \text{'work term'} \quad \text{.......................................(3.24)}$$

And the first law of black hole mechanics as,

$$dM = \frac{\kappa}{8\pi} dA + \text{'work term'} \quad \text{.......................................(3.25)}$$

Comparing (3.24) and (3.25) we see that if A plays the role of entropy S then κ plays the role of temperature T,

i.e. $\kappa\, dA \sim T\, dS$ ....(3.26)

Also it can be shown that the κ is constant across the event horizon surface. So we have an expression of zeroth law analogous to the zeroth law of thermodynamics.

Finally, there is the third law. For the extreme case we have,

$a^2 + Q^2 = M^2$ i.e. $\frac{J^2}{M^4} + \frac{Q^2}{M^2} = 1$ ....(3.27)

Then κ vanishes, although A does not vanished. This corresponds to absolute zero. It is the limiting case of an object which still possesses an event horizon. But the 'cosmic censorship hypothesis' implies the unattainability of 'absolute zero', $\kappa = 0$, so it plays the role of the third law.

CHAPTER FOUR
BLACKHOLE ENTROPY

4.1 Entropy:

Entropy is, in a sense, a measure of a system's energy that is unavailable for work or a measure of the disorder of a system. This quantity was first introduced by the German physicist Rudolf Clausius in 1850 as the amount of heat reversibly exchanged at a temperature T. When heat is added to a system held at constant temperature then the change in entropy depend to the change in energy, pressure and volume. Entropy undoubtly plays a major role in thermodynamics and statistical mechanics. It is also the most characteristics extensive parameter in thermodynamics;
namely when it is expressed in terms of other extensive parameters-it basically tells us the physical properties that underlines the system. Entropy is defined by the second law of thermodynamics and it enters the first law of thermodynamics to complete the differential representation of the internal energy, namely

$$dU = T\,dS - P\,dV + \mu'\,dN \qquad (4.1)$$

We can also expressed the entropy as a thermodynamic potential as

$$dS = \frac{1}{T}dU + \frac{P}{T}dV - \frac{\mu'}{T}dN \qquad (4.2)$$

where U is the internal energy, P is the pressure, V is the volume and N is the total number of molecules. Equation (4.2) is just the differential form of the entropy and we observe that , if the dependency of the entropy $S(U,V,N)$ on the variables U,V,N is known then complete knowledge of all the thermodynamic parameters is obtained.
Furthermore, the entropy tells us that for isolated systems (where $dQ_{reversible} = 0$) in equilibrium

$$dS = 0 \Leftrightarrow S = S_{\max imum} \qquad (4.3)$$

and for irreversible processes

$$dS > 0 \qquad (4.4)$$

So in words it says that the state of equilibrium is defined as the state of maximum entropy.
In statistical mechanics ,the definition of entropy was developed by Ludwig Boltzmann in 1870s by analyzing the statistical behavior of the microscopic components of the system. He shows that this definition of entropy is equivalent to the thermodynamic entropy to within a constant number known as Boltzmann's constant. In this definition , entropy is essentially a

measure of the number of ways in which a system may be arranged, often taken to be a measure of 'disorder'. Specifically, this definition describes the entropy as being proportional to the logarithm of the number of possible microscopic configurations of the individual atoms and molecules of the system which could give rise to the observed macroscopic state of the system. The constant of proportionality is the Boltzmann constant.

The most famous equation of statistical thermodynamics, the entropy of a system in which all states of number Ω , is given by

$S = k_B \ln \Omega$ ....(4.5)

where k_B is Boltzmann's constants. The microstate Ω is a function of the macrostate i.e. $\Omega(U,V,N)$. Hence entropy is a function of the variables U, V , and N .

The equation (4.5) is very important for it provides the basic connection between macroscopic thermodynamics (entropy) and statistical microscopic physics (number of microscopic states). We see that, $S = 0$ when $\Omega = 1$, thus there is only one exact microstate, hence no disorder – and no entropy is created.

4.2 Entropy in blackhole physics:

John Wheeler wrote in his book [89] about the genesis of the discovery of the concept of blackhole entropy by Jacob Bekenstein:

> " One afternoon in 1970 Bekenstein – then a graduate student and I were discussing blackhole in my office in Princeton's Jawdin Hall. I told him the concern I always feel when a hot cup of tea exchange heat energy with a cold cup of tea. By allowing that transfer of heat I do not alter the energy of the universe, but I do increase its microscopic disorder, its information loss, its entropy. The entropy of the world always increase in an irreversible process like that.
>
> " The consequence of my crime, Jacob, echo down to the end of time," I noted. " But if a blackhole swims by, and I drop the teacups into it, I conceal from all the world the evidence of my crime. How remarkable!"

> Bekenstein , a man of deep integrity, takes the lawfulness of creation as a matter of utmost seriousness. Several months later , he came back with a remarkable idea. " You don't destroy entropy when you drop those teacups into the blackhole. The blackhole already has entropy , and you only increase it!"
>
> Bekenstein went on to explain that the surface area of a blackhole is not only analogous to entropy, it is entropy, and the surface gravity of a blackhole (measured, for example, by the downward acceleration of a rock as it crosses the horizon) is not only analogous to temperature, it is temperature. A blackhole is not totally cold......".

Blackhole entropy and generalized second law of thermodynamics were introduced by Bekenstein in 1972 [5,3,90]. The area of a classical blackhole horizon cannot decrease in any process [63] is reminiscent of the second law of thermodynamics. Also blackhole mechanics analogs of the zeroth, first and third laws of thermodynamics[1] with surface gravity being analogous to temperature, horizon area with entropy and mass with internal energy. This analogy strongly supports the Bekenstein's proposal though some scientists take it only a mathematical curiosity.

Bekenstein was the first man who go to beyond mere analogizing and propose that blackhole actually carry an entropy. He suggests that by the area of blackhole event horizon we can measure how much entropy of a blackhole could have, although this idea contradict with thermodynamic entropy where entropy is related to the volume.

Bekenstein gives a short list of the predecessors of blackhole entropy[91] such as Christodoulou's irreducible mass[7], Wheeler's suggestion of a demon who violates the second law with the help of a blackhole[92] , Penrose and Floyd's observation that the event horizon area tends to grow [6], and Hawking's area theorem[8]. Upon the basis of their work Bekenstein concluded that the entropy of a blackhole must be proportional to the horizon area i.e.

$S_{bh} \propto A$ .... (4.6)

Now, since area is length squared and entropy is dimensionless quantity , so it would be reasonable that the proportionality constant in (4.6) must be of inverse squared length. For the universally validity of the relation (4.6) , this constant should be independent of blackhole parameters. Also it should not depend on interaction constants of non-gravitational interactions. Thus, the only available fundamental length in this case is the Planck length[93], $l_p \equiv (\frac{G\hbar}{c^3})^{\frac{1}{2}} \sim 10^{-33}$ *c.m.* Thus equation (4.6) gives

$$S_{bh} = \eta \frac{A}{l_p^2} \qquad (4.7)$$

where η is dimensionless constant of $O(1)$. According to Bekenstein, the horizon area divided by Planck's length squared is really an entropy , not just an analog of entropy. Although this proposal met initially some opposition[1,92,93], but accepted widely after Hawking's demonstration [4] that blackhole can emit particles. By the end of the 1970's most of the researchers agree with Bekenstein that at least a quasistatically and semi-classically evolving one, blackhole should carry an entropy.

4.3 Evidence for blackhole entropy:

(a) To satisfy the second law of thermodynamics is to admit that blackholes should have entropy. If blackhole does not carry entropy then it is possible to violate the second law of thermodynamics by throwing objects into the blackhole. The increase of the entropy of the blackhole more than compensates for the decrease of the entropy carried by the objects that was swallowed.

(b) Usually blackholes are formed by the collapse of a quantity of matter or radiation , both of which carry entropy. Also, the blackholes interior and contents are veiled to an exterior observer. Thus a thermodynamic description of the collapse from that observer's viewpoint cannot be based on the entropy of that matter or radiation because these are unobservable. Associating entropy with the blackhole provides a handle on the thermodynamics.

(c) A blackhole is characterized by only three parameters, namely its mass , angular momentum and electric charge. For any specific choice of these parameters one can imagine many scenarios for the blackhole formation. Thus there are many possible internal states corresponding to that blackhole. In thermodynamics one meets a similar situation: many internal microstates of a system are all compatible with the one observed macrostate. Thermodynamic entropy quantifies the said multiplicity. Thus by analogy one needs to associate entropy with a blackhole.

(d) If matter or radiation falls into blackhole, then the event horizon of blackhole prevents an external observer to receiving any information about the blackhole. Thus a blackhole can be said to hide information. In ordinary physics entropy is a measure of missing information. Hence it makes sense to attribute entropy to a blackhole.

4.4 Expression for blackhole entropy:

Let us start with the most general blackhole in general relativity i.e. the Kerr-Newman blackhole in which the area of the event horizon is given by Smarr 1973[95],

$$A = 4\pi(r_+^2 + a^2) = 4\pi[2M^2 - Q^2 + 2M^2(1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4})^{\frac{1}{2}}] \quad \text{.........................(4.8)}$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$$

with $Q^2 < M^2, \quad J^2 < M^4, \; a = \frac{J}{M}, G = c = 1$

From equation (4.8) one can obtain the incremental formula

$$dM = \frac{\kappa}{8\pi} dA + \Omega\, dJ + \Phi\, dQ \quad \text{..................................(4.9)}$$

where κ, Ω, Φ are defined by

$$\kappa = \frac{2\pi(r_+ - r_-)}{A}, \qquad \Omega = \frac{\delta M}{\delta J}, \; \Phi = \frac{\delta M}{\delta Q} \; .$$

For Schwarzschild blackhole equation (4.9) takes the form

$$dM = \frac{\kappa}{8\pi} dA \quad \text{......................................(4.10)}$$

The equation (4.9) or (4.10) are familiar as the first law of blackhole mechanics and are similar to the first law of thermodynamics

$$dU = T\, dS - PdV \quad \text{.............................(4.11)}$$

where the second term represent the work done on the system.

By the analogy between the first law of blackhole mechanics and the first law of thermodynamics and on the basis of Hawking's area theorem, Bekenstein made the bold proposal that a blackhole should have an entropy S_{bh}, proportional to the area of its event horizon i.e.

$$S_{bh} = const. \times A \quad \text{.........................(4.12)}$$

Equation (4.12) is a relation between a thermodynamic quantity and a geometric measure, is so striking that it demands an interpretation. On the basis of Shannon's information theory and Brillouin's classic work relating it to thermodynamics, Bekenstein proposed an information theoretic explanation for S_{bh}.[96]

Consider, some ideal gas in a container is compressed isothermally. Then it is well known that the thermal entropy of the gas certainly decreases due to the compression. However , the information about the internal configuration of the gas increases because after the compression the molecules of the gas are more localized than before compression. In fact, according to Brillouin, the increase in information

$\Delta I = -\Delta S$, where , ΔS is the decrease in entropy. So , it follows that the entropy measures lack of information about the internal configuration of the system. If P_n is the probability of occurrence for the nth state then the entropy associated with the system is given by Shannon's formula,

$$S = -\sum_n P_n \ln P_n \quad \text{..(4.13)}$$

The smallest unit of information is the binary bit, with $n = 2$ and $P_n = \frac{1}{2}$; this corresponds to a maximum entropy of $\ln 2$, which might be taken to be a unit of entropy.

The blackhole entropy S_{bh} arises due to the lack of information about the nature of the gravitational collapse. According to the 'no hair' theorem, the post collapse configuration i.e. the blackhole is completely characterized by three parameters mass M, electric charge Q and angular momentum J which encodes in an unknown way the diverse set of events occurring during the collapse , just as a thermodynamic system characterized by a few number of quantities like pressure, volume, temperature etc. which encode the microstate of the system. So the blackhole entropy is not to be regarded as the thermal entropy inside the blackhole horizon. As Bekenstein remarks " In fact, our blackhole entropy refers to the equivalence class of all blackholes which have the same mass, charge and angular momentum, not to one particular blackhole."[3]

If we set $G = c = 1$, then from equation (4.7) we can obtain,

$$S_{bh} = \frac{\eta}{\hbar} A \quad \text{.................................(4.14)}$$

According to Bekenstein, the appearance of $\hbar$ in the expression for blackhole entropy is just like many other formulas for entropy of many thermodynamic system that are conventionally regarded as classical, for example, the Boltzmann ideal gas. Thus the appearance of $\hbar$ is " a reflection of the fact that the entropy is, in a sense , a count of states of the system and the underlying states of any system are always quantum in nature. It is thus not totally unexpected that η appears in (4.14). These observations also suggest that it would be somewhat pretentious to attempt to calculate the precise value of the constant without a full understanding of the quantum reality which underlies a 'classical' blackhole."[3]

In general, Bekenstein assume that the blackhole entropy S_{bh} , is a monotonically increasing function of its rationalized area;

$$S_{bh} = f(\alpha) \quad \text{.............................(4.15)}$$

where $\alpha = \frac{A}{4\pi}$ is the rationalized area.

Using Christodoulou's techniques, Bekenstein argues that the minimum increase in rationalized area of the blackhole due to an infalling particle of rest mass μ and radius b is given by

$$(\Delta\alpha)_{min.} = 2\mu b \quad \text{.............................(4.16)}$$

Now in order to make $(\Delta\alpha)_{min.}$ smaller by making b smaller but not than the particle's Compton wavelength $\hbar\mu^{-1}$ or than its gravitational radius 2μ , which ever is the larger. The Compton wave length is the larger for $\mu \le (\frac{\hbar}{2})^{\frac{1}{2}}$ and the gravitational radius is the larger for $\mu > (\frac{\hbar}{2})^{\frac{1}{2}}$. In the first case $(\Delta\alpha)_{min.} = 2\hbar$ and in the second case $(\Delta\alpha)_{min.} = 4\mu^2 > 2\hbar$. Thus $(\Delta\alpha)_{min.} = 2\hbar$, as is indeed the case for an elementary particle. This then also quantifies the minimum loss of information due to the particle entering the blackhole horizon. Recalling now that the minimum loss of information is a binary bit corresponding to an increase in entropy of $\ln 2$, one sets [3],

$$(\Delta S_{bh})_{min.} = \frac{df}{d\alpha}(\Delta\alpha)_{min.} = 2\hbar\frac{df}{d\alpha} = \ln 2 \quad \text{.................................(4.17)}$$

By integrating equation (4.17) one can obtain,

$$f(\alpha) = \frac{\alpha}{2\hbar}\ln 2 \quad \text{...(4.18)}$$

Including all factors G, c and k_B, the Bekenstein formula for blackhole entropy, in conventional units, is

$$S_{bh} = \frac{1}{8\pi G\hbar} \ln 2 \, k_B \, c^3 \, A \ldots\ldots(4.19)$$

From equation (4.19) we see that the entropy of the blackhole is enormous. The large numerical value of the blackhole entropy indicates the highly irreversible character of the process of blackhole formation . The temperature of the blackhole , T_{bh}, can be defined in analogy with the temperature in thermodynamics; $T^{-1} = (\frac{\delta S}{\delta U})_V$; Here $T_{bh}^{-1} = (\frac{\delta S_{bh}}{\delta M})_{J,Q}$

Using equation (4.19) we can obtain the temperature of the blackhole as

$$T_{bh} = \frac{2\hbar}{8\pi \ln 2} \kappa \ldots\ldots(4.20)$$

with κ is the surface gravity of the blackhole, a geometric quantity which remains constant over the event horizon.

4.5 Evidence for $S_{bh} = f(A)$:

In ordinary thermodynamics, entropy and temperature are definite function of energy, volume and pressure. If we consider a thermodynamical system in equilibrium including blackholes and its surroundings then the blackhole entropy S_{bh} and temperature T must be a function of blackholes macroscopic parameters. According to 'no hair' theorem these parameters are only mass M, angular momentum J and electric charge Q.

Applying the first law of thermodynamics including blackholes and its surrounding we have,

$$dM = TdS_{bh} + dW \ldots\ldots(4.21)$$

where dW is the work term on the blackhole.

For the most general type of blackhole i.e. Kerr-Newman blackhole, the work term dW takes the form,

$$dW = \Phi\, dQ + \Omega\, dJ \ldots\ldots(4.22)$$

where Ω is the angular velocity and Φ is the electric potential defined by,

$$\Omega = \frac{\delta M}{\delta J} = \frac{4\pi a}{A} \ldots\ldots(4.23)$$

$$\Phi = \frac{\delta M}{\delta Q} = \frac{4\pi r_+ Q}{A} \ldots\ldots(4.24)$$

with $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$ = radius of event horizon,

$a = \frac{J}{M}$, $A = 4\pi(r_+^2 + a^2)$ = area of the event horizon.

Now using equation (4.22) in (4.21) we obtain,

$dM = T\,dS_{bh} + \Phi\,dQ + \Omega\,dJ$

or, $T\,dS_{bh} = dM - \Phi\,dQ - \Omega\,dJ$ ....(4.25)

Again differentiating horizon area A with respect to M, J and Q we obtain the first law of blackhole mechanics as,

$\frac{\kappa}{8\pi} dA = dM - \Phi dQ - \Omega dJ$ ....(4.26)

where κ is the surface gravity defined by,

$\kappa = 8\pi(\frac{\delta M}{\delta A})_{J,Q} = \frac{4\pi\sqrt{M^2 - Q^2 - a^2}}{A} = \frac{2\pi(r_+ - r_-)}{A}$ ....(4.27)

with $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$.

Comparing equation (4.25) and (4.26) we get,

$T\,dS_{bh} = \frac{\kappa}{8\pi} dA$ ....(4.28)

From equation (4.28) we can infer that the blackhole entropy S_{bh} is a function of its horizon area A ;

$S_{bh} = f(A)$ ....(4.29)

On the basis of the argument that the two blackholes with the same area must have the same entropy since otherwise one can violate the second law by Penrose processes[6,97] , Andrew Gould[98] obtain the same result as equation (4.29).

Although the equation (4.29) derived on the consideration of Kerr-Newman blackhole but it seems that it can be applied to other types of blackholes. For example, a (2+1) dimensional BTZ blackhole obeys the standard first law of thermodynamics (4.25) augmented by an additional work term, $-Pd(2\pi R)$, where P is the surface pressure at the boundary of the cavity of radius R [99]. Accordingly , the same arguments which applied to prove that the relation $S_{bh} = f(A)$ for the Kerr-Newman blackhole, may be used now to prove that the entropy of the BTZ blackhole must be some function of its horizon area, and indeed, semi-classical calculations yields a linear entropy to horizon area relation[100].

Recently Vaz and Witten[101] show that in the frame work of canonical quantum gravity the entropy of a charged blackhole turns out to be the difference between the outer and inner horizon areas, conflicting with the result $S_{bh} = f(A)$. Vaz and Witten explain this disagreement with the semi-classical result, as the product of neglecting the effect of back reaction to any radiation emitted exterior to the blackhole. Another explanation involves the similar effect induced by emission of radiation in the interior of the blackhole, radiation which is undetectable by an exterior observer. However, as Vaz and Witten admit themselves, their result may be due to the too restrictive boundary conditions imposed on the wave functional in the interior of the blackhole(the wave functional is made to vanish beyond the inner horizon). However, they hedge this claim by pointing out that other boundary conditions are in effect unknown.

However , we define the temperature T of the blackhole,

$$T = \frac{\delta M}{\delta S_{bh}} = \frac{\kappa}{8\pi f'(A)} = \frac{\sqrt{M^2 - Q^2 - a^2}}{2A f'(A)} \qquad (4.30)$$

It is notable that the blackhole have a non-negative temperature, so $f'(A) > 0$ i.e.

$f(A)$ must be a monotonic non-decreasing function which supports Hawking's area theorem and Bekenstein's entropy of blackhole.

By the parallelism between the zeroth law of thermodynamics (temperature is constant over a system of equilibrium) and the zeroth law of blackhole mechanics(surface gravity is constant over the event horizon surface of a stationary blackhole)
we can infer that the blackhole temperature must be a definite function of the surface gravity. Therefore , from equation (4.30) we conclude that $f'(A)$ must be a constant.

Hence,

$$f'(A) = \xi = constant \qquad (4.31)$$

By integrating we obtain,

$$S_{bh} = f(A) = \xi A + \xi' \qquad (4.32)$$

If we impose the condition ,

$M = 0$ *i.e.* $A = 0, S_{bh} = 0$ then $\xi' = 0$ and so ,

$$S_{bh} = f(A) = \xi A$$

which shows that the entropy of blackhole is linear to its horizon area.

4.6 Interpretation of blackhole entropy:

Obviously, the term 'blackhole entropy' is an absurd term in blackhole physics.We all have been convinced by now by the fact that the blackholes carry an Bekenstein-Hawking entropy $S_{BH} = \frac{A}{4l_p^2}$, but its deeper meaning has remained mysterious.There are many interpretations have been suggested to explain the blackhole entropy, including some novel and profound ideas such as [91,79,102,103,104,105,106,107].On the basis of quantum information and recent analysis of Hawking radiation, Baocheng Zhang, Qing-yu Cai, Ming-sheng Zhan and Li You [104] interprete blackhole entropy as the measure of uncertainty about the information of the blackhole forming matter's pre-collapsed configurations, self-collapsed configurations and inter-collapsed configurations. They applied it to several circumstances, including the formation of a blackhole, blackhole coalescence and a common matter dropped into a blackhole.

Jarmo Makela and Pasi Repo [107] suggests that the blackhole entropy can be interpreted in two possible ways. The first is, there is the conservative view that the entropy of the blackholes may be understood as a result of a huge degeneracy in the mass eigenstates of the whole blackhole spacetime. The degeneracy of the eigenstates might somehow, in a still unexplained manner, allow one to include the degrees of freedom of the collapsed matter, but this view contradicts with the 'no hair' theorem.The second view -called the external point of view- is that the entropy of blackhole is , quite simply, caused by the fact that the interior region of the blackhole spacetime is separated from its exterior region by a horizon. Because of that , for an external observer, it is justified that the statistical mechanics of blackhole is, the statistical mechanics of blackhole's exterior region. On the basis of this point of view , one can obtain Bekenstein-Hawking entropy of the blackhole without assuming any degeneracy in the mass eigenstates of the blackhole. This result is not contradictory with the 'no hair' theorem but allows a complete loss of information, since the degrees of freedom of the matter, except the total mass M, have vanished. Thus the two points of view to the interpretation of blackhole entropy, of which neither is quite completely satisfactory.

It is already given that , according to statistical mechanics, the entropy is a measure of the multiplicity of microstates that hide behind one particular macrostate. A special case of this is Boltzmann's famous formula $S = k_B \ln \Omega$, where Ω stands for the number of equally probable microstates of a particular macrostate. Since blackhole entropy plays a role quite analogous to that of ordinary thermodynamic entropy, e.g. it participates in the second law of thermodynamics, many have wondered what the microstates that are counted by the blackhole entropy are. Following [94], a partial list of interpretations of blackhole entropy is given below:

(a) Blackhole entropy counts the number of internal states of matter and gravity: According to classical 'no hair' theorem after the collapse, when a blackhole has settled down to a stationary state, its properties are determined by very few parameters observed by exterior observer namely mass M, angular momentum J and electric charge Q. Thus from this point of view , blackhole have only three degrees of freedom. Thus one can infer that the enormous amount of degrees of freedom and information of the collapsed matter are lost during the collapse. The entropy of the blackhole may be understood as a measure of information loss during the gravitational collapse, because there is a well-known relationship between entropy and information given by Brilliouin[108]; the decrease in information increase the entropy. This approach is purely quantum mechanical. According to quantum mechanics, all the information from the collapsing star is not able to reach to an observer exterior to the newly formed event horizon. In other words, all the microstates of the collapsing star cannot be measured by the external observer.This results to an increasing entropy. Bekenstein [3] who introduced the notion of blackhole entropy related it with " the measure of the inaccessibility of information (to an exterior observer) as to which a particular internal configuration of the blackhole is actually realized in a given state"(i.e. for a given value of M, J and Q). According to the " standard" interpretation these different internal states of a blackhole are related with different possible initial conditions which may result in the creation of a stationary blackhole with the same parameters M, J and Q [3]. In this approach the entropy of a blackhole is considered as the logarithm of the number of distinct ways that the blackhole might have been made[79]. Since the Bekenstein-Hawking entropy of blackhole is given by $S_{BH} = \frac{A}{4l_p^2}$, with Boltzmann's constant $k_B = 1$,so one might expect that there are $e^{S_{BH}}$ microstates corresponding to the same macrostate of the blackhole. This point of view has been illustrated by Frolov and Novikov[102].

(b) Blackhole entropy as an entanglement entropy:

The statistical entropy measure the unavailable information (missing) about the system, which one might acquire by knowing the system better through microscopic measurement. Now as mentioned earlier the blackhole and due to the presence of event horizon one has no access to the inside region, affords an objective way of coarse-graining, namely neglect whatever is inside the horizon, an entropy based on this type of coarse-graining is known as the entanglement entropy.

Entanglement entropy was introduced very early in relativity to understand the Unruh effect as resulting from ignoring the states beyond the Rindler horizon. The interpretation of blackhole entropy in terms of quantum entanglement entropy first proposed by Bombelli, Koul, Lee and Sorkin[103] (BKLS) .The observation that was made by BKLS was that the exterior region of the blackhole has a well defined autonomous dynamics,- no information is fed into it from the inside horizon-one can expect a second law to apply to an entropy defined exclusively in it.

This type of entropy when calculated turned out to be divergent due to the entanglement between values of the quantum field just inside and just outside the horizon, and if no cutoff were introduced the entropy would diverge. However, BKLS shows that if a cutoff is introduced the result would come out to be proportional to the area of the horizon with proportionality constant quadratic in the cutoff. Srednicki [109] rediscovered the idea of BKLS and pointed out that the global vacuum states of a scalar field in flat spacetime, when restricted to the exterior region of an imaginary sphere , is in a mixed state there. The density matrix of this mixed states arises from tracing out those parts of the global state that reside inside the sphere; its entropy is evidently related to the unknown information about the sphere's interior. This entropy is nonvanishing only because the exterior state is correlated with the interior one. In the sphere's case the quantum entanglement entropy comes out to be proportional to the sphere's surface area with a coefficient which diverges quadratically in the high frequency cutoff [109].

BKLS also gave reasons for relating at least part of the blackhole entropy to entanglement entropy of the state outside blackhole. In particular they pointed out that whereas for an ordinary " black box" situation the emergency of entanglement entropy out of a pure state is to a large extent a matter of choice for the observer, for the blackhole case the horizon's presence makes its emergency mandatory.

The entanglement entropy has been lately considered by several authors with the same conclusion. Susskind, Thorlacius and Uglum [110] explained entanglement entropy as the relation between entanglement and radiation entropy. Holzhey [111] and Callan and Wilczek [112] have made use of clever techniques for computing it, concluding with BKLS and Srednicky that a plane boundary in Minkowski spacetime, when the quantum state beyond it is ignored, gets ascribed entanglement entropy proportional to the area of the boundary with an ultraviolate quadratically divergent coefficient. Kabat and Strassler[113] further show that the density operator
in question is thermal irrespective of the nature of the field. Holzhey, Larsen and Wilczek [114] explore a method to regularize the divergence in conformal field theories.

Bekenstein[91] argued that the entanglement entropy is operationally finite(at least in flat spacetime). The claim was that , it is untrue , in general, that one knows nothing about the interior states. For example by knowing the size of the internal region one can set a bound on the energy which one has to trace out and hence providing cutoff and making the entropy operationally finite.

In trying to identify the entanglement entropy as the blackhole entropy one ends up with the conclusion that its dependent on the number of species of fields that exists in nature, since each field must make its contribution to the entanglement entropy. Yet $S_{BH} = \frac{A}{4}$ says nothing about the number of species!

(c) Blackhole entropy counts the number of horizon gravitational states:

It has been suggested that the sought for states are the states of the gravitational degrees of freedom residing on the blackhole's horizon. An example of this approach is a calculation by Carlip[115]based on the group of symmetries at the horizon. It reproduces the formula $S_{BH} = \frac{A}{4l_p^2} = \frac{c^3 A}{4G\hbar}$.

(d) Blackhole entropy is a conserved quantity connected with coordinate invariance of the gravitational action / Blackhole entropy as a Noether charge:

This abstract approach has been championed by Wald[116] as a road to blackhole entropy in more general theories of gravity than general relativity.

Wald derived a general formula for the entropy of a stationary blackhole based on a Lagrangian derivation of the first law of blackhole mechanics. The result of [116] apply to blackholes with bifurcate Killing horizons in any diffeomorphism invariant theory in any spacetime dimension. Wald finds that the entropy of the blackhole is given by ,

$$S_{BH} = 2\pi \oint_{\Sigma} Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.33)$$

where Σ is the bifurcation surface of the Killing horizon. The (n-2) form Q is the 'Noether charge' associated with the Killing field, normalized so as to have unit surface gravity. T. Jacobson, G. Kang and R. C. Myers [129] obtain the same result for the entropy of blackhole if Q is integrated over any cross-section of the horizon.

[Technically, blackhole entropy is the Noether charge of the diffeomorphism symmetry. This notion reproduces the formula $S_{BH} = \frac{A}{4l_p^2} = \frac{c^3 A}{4G\hbar}$ when the gravitational action is of first order in the curvature, but gives a modified formula for higher order gravity theories.]

(e)Blackhole entropy is thermal entropy of the gas of quanta constituting the thermal atmosphere of the blackhole:

In this approach the origin of the blackhole entropy is related to the properties of the physical vacuum in strong gravitational fields. There are always zero-point fluctuations of physical fields in a vacuum state. An observer who is at rest with respect to the horizon sees this vacuum excitations as a thermal atmosphere around a blackhole . The first attempts to relate the Bekenstein-Hawking entropy to the thermal atmosphere were made by Thorne and Zurek [79] and by 't Hooft[117]. Moreover, 't Hooft introduced the concept of a "brick wall" to keep the said atmosphere from contacting the horizon and thus making the entropy infinite. This approach recovers the proportionality of entropy to the horizon area, but the coefficient has to be chosen by hand.

(f) Blackhole entropy counts the number of states or excitations of a fundamental string:

Strings in string theory have a variety of excitations , so there is a multitude of string states. Therefore , a string has entropy, which turns out to be proportional to its mass. This is quite in contrast with blackhole entropy. However, an argument by Bowick, Smolin and Wijewardhana [118] suggests that by adiabatically (i.e. sufficiently slowly) reducing the string coupling constant g , it is possible to shrink a blackhole's size as well as to reduce its mass (while keeping its entropy constant) until eventually it gets to be the size of the string length scale l_s when the blackhole should not be distinguishable from a string. At the corresponding value of g , string and blackhole entropy are quite similar[119]. This has been taken to mean that there is a one-to-one correspondence between blackhole and string states, where both entities have the same entropy[120]. This picture has been corroborated in the context of five-dimensional extreme blackholes[121]. Hence blackhole entropy can be understood in terms of string entropy.

(g)Blackhole entropy is equivalent to the thermal entropy of the radiation residing on the boundary of the spacetime containing the blackhole:

The Ads/CFT correspondence is a mapping between gravitational degrees of freedom of a certain spacetime and the matter (or field) degrees of freedom residing on its boundary. In particular, certain string theories in five dimensional Anti-deSitter(Ads) spacetime are so mapped to conformal field theories on the corresponding spacetime's four dimensional boundary which bears resemblance to Minkowski spacetime. Witten[122] has shown that the entropy of a blackhole residing in the bulk Anti-deSitter spacetime equals that of thermal radiation of the fields residing on its boundary.

4.7 Blackhole entropy problem:

In classical theory, blackholes are considered to absorb matter, but emit nothing. Due to 'no hair' theorem, a blackhole is regarded as a candidate of the final state of matter. But in physics, it is generally believed that the objects which has absorbing property, must have the radiating property also. However, in 1974 Hawking [4] argued that the blackholes emit thermal radiation by taking account of quantum effect of fields. This essentially establishes the belief that the blackholes are thermodynamical existence, which have temperature $T = \frac{\hbar\kappa}{2\pi}$ and an entropy $S_{BH} = \frac{A}{4l_p^2}$ where T is the temperature of the blackhole , κ is the classically defined surface gravity,A is the horizon area and l_p is the Planck length. Usually thermodynamical system have statistical dynamical description using entropy concept. However , the statistical dynamical picture of blackholes has not been established yet. This is called the blackhole entropy problem.

In a sense, thermodynamical entropy S is defined by the response of the free energy F of the system on the changes of its temperature i.e.

$$dF = -S\,dT \quad \text{...(4.34)}$$

Upon the basis of this definition Bekenstein and Hawking obtain the entropy formula $S_{BH} = \frac{A}{4l_p^2} = \frac{c^3 A}{4G\hbar}$ for blackholes. Statistical mechanical entropy S_{sm} is defined as,

$$S_{sm} = -Tr(\rho \ln \rho) \quad \text{.................................}(4.35)$$

where ρ is the density matrix describing the internal state of the system under consideration. Also it is possible to introduce the informational entropy S_I by counting different possibilities to prepare a system in a final state with given macroscopical parameters from different initial states,

$$S_I = -\sum_n P_n \ln P_n \quad \text{...................................}(4.36\ )$$

where P_n being the possibilities of different initial states. In standard case all three definitions give the same result. But Frolov[123] show that the S_{BH} does not coincide with the statistical mechanical entropy of a blackhole.

According to statistical mechanics, entropy is the logarithm of the number of microscopically different states available for given values of the macroscopic parameters. From this point of view, the question arises as, are there microscopic internal degrees of freedom that are responsible for the Bekenstein-Hawking entropy S_{BH} ? This is the question that physicist were trying to answer for almost 35 years.

The main reason why this question is fundamental is because it goes beyond the blackhole physics itself. Its answer may give important insight into the as yet mysterious nature of quantum gravity. To see this let us start with a simple estimation and consider a static suppermassive blackhole of mass M of the order of 10^9 solar masses. It is believed that this type of blackhole exist in the center of the galaxies. By taking into account $A = 16\pi G^2 M^2$ and $S_{BH} = \frac{A}{4G}$, one can obtain the entropy of such a blackhole is of the order of 10^{95}. This is seven order of magnitude larger than the entropy of the other matter in the visible part of the universe[124]. What makes matters even worse is that in the classical theory, a blackhole is nothing but an empty spacetime with a strong gravitational field. Thus , an explanation of the Bekenstein-Hawking entropy is one of those problem which cannot be solved in classical gravity theory.If we consider the blackhole horizon surface is covered by cells of a Planckian size $L_{P_l} \sim \sqrt{G}$, then according to the relation $S_{BH} = \frac{A}{4G}$, S_{BH} is of the same order of magnitude as the logarithm of the number of different ways to distribute two signs $'+'$ and $'-'$ over these cells[125]. The appearance of the Planck scale in this estimation indicates that a reasonable microscopical explanation problem of Bekenstein-Hawking entropy must be based on the quantum gravity. Bekenstein argued that $\Omega = e^{S_{BH}}$ must in some sense be the number of " quantum mechanically distinct internal states" that a blackhole could have, corresponding to its classically observed external parameters[79]. The question now arises; after the collapse of matter, are the degrees of freedom contained in the matter fields somehow encoded into the quantum states of the blackhole spacetime itself, or have they vanished altogether, leaving no trace whatsoever ? Of course , it is natural to claim that they are encoded into the quantum states of spacetime itself such that there is a vast $e^{S_{BH}}$ - fold degeneracy in the quantum states of the blackhole. This leads us to a conclusion that the total number of unknown quantum states of the blackhole must be enormous, too. Thus, from a quantum mechanical point of view , the number of the physical degrees of freedom of the blackhole is not limited to just few parameters. Obviously, the contradiction between quantum and classical blackholes is that the number of the physical degrees of freedom of the classical blackhole is three, whereas the number of the physical degrees of freedom of the quantum blackhole is enormous. The problem with this contradiction is that it is not quite clear how, starting from general relativity, quantization itself might bring along a huge number of additional degrees of freedom [107].Also, at present there is no satisfactory statistical mechanical derivation of the entropy S_{BH} and the physical nature of the blackholes " internal states" has remained a puzzle. Three answers to this puzzle have been proposed in [79]; (i) Gerlach's [126] view of Hawking radiation as produced by zero-point fluctuations on the surface of the star that collapsed to form the blackhole and his calculation that the number Ω_{ZP} of zero-point fluctuation modes that give rise to the Hawking radiation of a freely evaporating Schwarzschild blackhole satisfies $\ln \Omega_{ZP} \cong 280 S_{BH}$. (ii) York's [126] view of Hawking radiation as produced by the blackhole's "quantum ergosphere" of thermally excited gravitational quasinormal modes, and his conclusion that the number of ways Ω_{qe} that this quantum ergosphere can be excited and reexcited, during the evaporation of a Schwarzschild blackhole into a surrounding radiation bath, satisfies $\ln \Omega_{qe} \cong 1.10617 S_{BH}$. (iii) A view implicit in the writings of Bekenstein and Hawking that $\Omega = e^{S_{BH}}$ might be the number of quantum mechanically distinct ways that the blackhole could have been made by infalling quanta(particles).

At the present moment, the Bekenstein-Hawking entropy obtained by counting of string(D-brane) states but this calculations essentially use supersymmetry and are mainly restricted to extreme and non-extreme blackholes. Also it remains unclear why the entropy of the blackhole is universal and does not depend on the details of the theory at Planckian scales. It is noted that the thermodynamics of blackhole follows from the low energy gravitational theory. That is one can expect that only a few fundamental properties of quantum gravity but not its concrete details are really important for the statistical mechanical explanation of blackhole entropy.

Now, we would like to set of questions to which complete answers still lack. Some questions are given below in order:

(i) How does General relativity know about the blackhole thermodynamics? [126]

(ii) Why should blackholes at all have entropy ? [127]

(iii) Is the analogy between blackhole thermodynamics and the standard thermodynamics complete ? [123]

(iv) Is blackhole entropy S_{bh} real or subjective ? [59]

(v) Where does it appear- on or near the horizon or deep in the hole ? [59]

(vi) At what stage in the blackhole's evolution is it created – immediately upon formation by gravitational collapse, or only gradually over the long course of evolution ? [59]

(vii) In what sense is $T = \frac{\hbar\kappa}{2\pi}$ the temperature of the blackhole ? [127]

(viii) What becomes of the standard second law of thermodynamics in presence of blackhole ? [127]

(ix) What are the microstates whose counting would yield the area law for blackhole entropy ? [127]

(x) What is the dynamical mechanism that makes S_{bh} a universal function, independent of the hole's past history or detailed internal condition? [59,128]

(xi) Do there exist internal degrees of freedom of a blackhole which are responsible for its entropy ? [123,126]

(xii) Where are the microscopic degrees of freedom responsible for blackhoe entropy located ? [128]

(xiii) Can S_{bh} be derived from quantum mechanical consideration ? [59]

(xiv) Due to the Hawking radiation (blackhole evaporation), what happens to S_{bh} after the blackhole has evaporated ? Will all the information disappear after the evaporation ? [59]

(xv) Is there any information loss in blackhole dynamics ? [126]

(xvi) Is blackhole entropy similar to that of ordinary entropy i.e. the logarithm of a count of internal blackhole states associated with a single blackhole exterior ? [91]
(xvii) Is blackhole entropy the logarithm of the number of ways in which the blackhole might be formed ?[91]
(xviii) Is blackhole entropy the logarithm of the number of horizon quantum states ? [91]
(xix) Does it stand for information lost in the transcendence of the hallowed principle of unitary evolution ? [91]
(xx) Is it possible to apply the statistical mechanical and informational definitions of the entropy to blackholes and how are they related with the Bekenstein-Hawking entropy ?[123]
(xxi) How does a pure state evolve into a mixed(thermal) state ? Is there a information loss due to the formation of blackhole and Hawking radiation process ? Does the usual quantum mechanics need to be modified in the context of blackhole ?[128]
(xxii) Can quantum theory of gravity remove the formation of spacetime singularity due to the gravitational collapse ? [128]
(xxiii) Unlike other thermodynamical systems, why is blackhole entropy non-extensive ? i.e. Why S_{bh} is proportional to the area and not volume ? [128]
(xxiv) Why is the blackhole entropy large ? [128]
(xxv) How S_{BH} concords with the standard view of the statistical origin ? What are the blackhole microstates ? [128]
(xxvi) Are there corrections to S_{BH} ? If there are, how generic are they ?[128]
At present , physicists answers some of the question mentioned above but not quite satisfactory . So these questions are somewhat embarrassing , because we do not know with our present knowledge how to answers them precisely. Neverthless, it is hopped that success in modern theory of gravity, e.g. , the quantum gravity or the string theory would be the key to answer-if not all-some of these open questions.

4.8 Bekenstein-Hawking entropy, Hawking temperature and other Intensive Parameters of Blackholes :

According to Bekenstein, blackhole entropy is proportional to the area of its horizon divided by Planck area. He suggested that the constant of proportionality is $\frac{\frac{1}{2}\ln 2}{4\pi}$ and if the constant of proportionality not exactly $\frac{\frac{1}{2}\ln 2}{4\pi}$, then it must be very close to it. Later, Hawking showed that the blackhole emit thermal Hawking radiation corresponding to a certain temperature. Using thermodynamic relationship between energy, temperature and entropy, he was able to confirm Bekenstein's conjecture and fix the value of the constant of proportionality at $\frac{1}{4}$. Using this value the blackhole entropy formula (4.6) takes the form,

$$S_{BH} = \frac{1}{4}\frac{A}{l_p^2} \qquad \text{.............................(4.37)}$$

The formula (4.37) is familiar as the Bekenstein-Hawking entropy formula of blackhole and the subscript BH stands for Bekenstein-Hawking entropy of blackholes. Including Boltzmann's constant and set $G = c = \hbar = \frac{h}{2\pi} = 1$, the equation (4.37) becomes,

$$S_{BH} = \frac{1}{4}k_B\, A \qquad \text{..........................(4.38)}$$

The mass formula which contains all the information about the thermodynamic state of the Kerr-Newman blackhole already given by the equation (4.8) and from that equation we can obtain,

$$M = \sqrt{\frac{1}{4}\left(\frac{A}{4\pi}\right) + \frac{4\pi}{A}\left(J^2 + \frac{Q^4}{4}\right) + \frac{1}{2}Q^2} \qquad \text{.......................(4.39)}$$

Using equation (4.38) with $k_B = \frac{1}{\pi}$ in (4.39) we have

$$M = \sqrt{\frac{S_{BH}}{4} + \frac{1}{S_{BH}}\left(J^2 + \frac{Q^4}{4}\right) + \frac{1}{2}Q^2} \qquad \text{............................(4.40)}$$

We will use this relation for obtaining certain thermodynamic properties of four familiar families of blackholes and then the theoretical BTZ blackhole.

4.8 (a) Schwarzschild blackhole:

Putting $J = Q = 0$ in the mass formula (4.40) we obtain the mass formula for Schwarzschild blackhole as,

$$M = \sqrt{\frac{S_{BH}}{4}} \quad \text{.................................(4.41)}$$

and from above equation we obtain the entropy of the Schwarzschild blackhole as,

$$S_{BH} = 4M^2 \quad \text{................................(4.42)}$$

The temperature of the Schwarzschild blackhole is given by,

$$T = \frac{\delta M}{\delta S_{BH}} = \frac{1}{8M} \quad \text{......................(4.43)}$$

We see from the equation (4.43) that the temperature of Schwarzschild blackhole is inversely varies with its mass i.e. the larger of the mass, the smaller of the temperature. Obviously the angular velocity and electric potential of the Schwarzschild blackhole is zero. The heat capacity of the Schwarzschild blackhole is given by,

$$C = \frac{\delta M}{\delta T} = \frac{\delta M}{\delta S_{BH}} \cdot \frac{\delta S_{BH}}{\delta T} = \frac{T}{\frac{\delta T}{\delta S_{BH}}} = \frac{T}{\frac{\delta^2 M}{\delta S_{BH}^2}} = -2S_{BH} \quad \text{......................(4.44)}$$

which is a negative quantity.

4.8 (b) Ressiner-Nordström blackhole:

Putting $J = 0$ in equ. (4.40) we obtain the mass formula for Reissner-Nordström blackhole as ,

$$M = \frac{\sqrt{S_{BH}}}{2}(1 + \frac{Q^2}{S_{BH}}) \quad \text{..........................(4.45)}$$

By solving equation (4.45), we obtain the entropy of the Reissner-Nordström blackhole as ,

$$S_{BH} = 2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2}} \quad \text{..........................(4.46)}$$

The temperature, electric potential and heat capacity of Reissner-Nordström blackhole are given by ,

$$T = \frac{\delta M}{\delta S_{BH}} = \frac{1}{4\sqrt{S_{BH}}}(1 - \frac{Q^2}{S_{BH}}) \quad \text{.............................(4.47)}$$

$$\Phi = \frac{\delta M}{\delta Q} = \frac{Q}{\sqrt{S_{BH}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.48)$$

$$C = \frac{T}{\frac{\delta^2 M}{\delta S_{BH}^2}} = \frac{2S_{BH}(S_{BH} - Q^2)}{3Q^2 - S_{BH}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.49)$$

The heat capacity of the Reissner-Nordström blackhole is a negative quantity when $S_{BH} > Q^2$ and both heat capacity and the absolute temperature is zero associated with the hole when $Q = \sqrt{S_{BH}}$, which is the extremal limit.

4.8(c) Kerr blackhole:

The mass formula for the Kerr blackhole is obtained by putting $Q = 0$ in equ. (4.40) which reads ,

$$M = \sqrt{\frac{S_{BH}}{4} + \frac{J^2}{S_{BH}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.50)$$

Solving for S_{BH} , we have

$$S_{BH} = 2M^2 + 2M^2\sqrt{1 - \frac{J^2}{M^4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.51)$$

The temperature of the Kerr blackhole takes the form,

$$T = \frac{\delta M}{\delta S_{BH}} = \frac{1}{4}\,\frac{(1 - \frac{4J^2}{S_{BH}^2})}{\sqrt{S_{BH} + \frac{4J^2}{S_{BH}}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.52)$$

The temperature vanishes when $S_{BH} = 2J$ or in terms of M and J it is $J = M^2$ i . e . for extremal case. The angular velocity of the hole is given by,

$$\Omega = \frac{\delta M}{\delta J} = \frac{J}{S_{BH}\sqrt{\frac{S_{BH}}{4} + \frac{J^2}{S_{BH}}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.53)$$

The heat capacity C is given by

$$C = \frac{T}{\frac{\delta^2 M}{\delta S_{BH}^2}} = -\frac{2S_{BH}(1 - \frac{16J^4}{S_{BH}^4})}{(1 - \frac{24J^2}{S_{BH}^2} - \frac{48J^4}{S_{BH}^4})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.54)$$

which also vanishes when $S_{BH} = 2J$.

4.8 (d) Kerr-Newman blackhole: The mass formula for the Kerr-Newman blackhole is already given by equ.(4.40) and from that equation we can obtain,

$$S_{BH} = 2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}} \quad \text{.........................(4.55)}$$

The temperature of the Kerr-Newman blackhole is given by,

$$T = \frac{\delta M}{\delta S_{BH}} = \frac{S_{BH}^2 - 4J^2 - Q^4}{8MS_{BH}^2} \quad \text{..................................(4.56)}$$

Angular velocity Ω is given by

$$\Omega = \frac{\delta M}{\delta J} = \frac{J}{MS_{BH}} \quad \text{...(4.57)}$$

From equation (4.57) we see that if the mass and entropy of the blackhole increases then the angular velocity of the blackhole decreases and vice versa. Also, as we expected that the blackhole's angular velocity is proportional to its angular momentum .

The electric potential of the Kerr-Newman blackhole is given by,

$$\Phi = \frac{\delta M}{\delta Q} = \frac{Q(S_{BH} + Q^2)}{2MS_{BH}} \quad \text{................................(4.58)}$$

From equation (4.58) we see that the electric potential is reduced by mass and entropy of the blackhole. The heat capacity of the Kerr-Newman blackhole is given by ,

$$C = \frac{T}{\frac{\delta^2 M}{\delta S_{BH}^2}} = \frac{8S_{BH}^2 M^2 (S_{BH}^2 - 4J^2 - Q^4)}{16S_{BH} M^2 (4J^2 + Q^4) - (S_{BH}^2 - 4J^2 - Q^4)^2} \quad \text{..................(4.59)}$$

4.8 (e) BTZ blackhole:

The equation of the event horizons of the BTZ blackhole is given by , $r_{\pm} = l\sqrt{\frac{M}{2}(1 \pm \Delta)}$ ....(4.60)

where $\Delta = \sqrt{1 - (\frac{J}{Ml})^2}$.

For simplicity we will use $l = 1$ and the equation (4.60) takes the form

$$r_{\pm} = \sqrt{\frac{M}{2}(1 \pm \Delta)} \quad \text{................................(4.61)}$$

and $\Delta = \sqrt{1 - (\frac{J}{M})^2}$.

For this particular blackhole if we use the Boltzmann's constant $k_B = \frac{2}{\pi}$,

then the entropy of the BTZ blackhole is given by,

$$S_{BH} = \frac{A}{2\pi} = \frac{L}{2\pi} = \frac{2\pi r_+}{2\pi} = r_+ = \sqrt{\frac{M}{2}(1+\Delta)} \quad \text{..........................(4.62)}$$

where L is the length of the horizon r_+.

From equation (4.62) we can obtain,

$$M = S_{BH}^2 + \frac{J^2}{4S_{BH}^2} \quad \text{...............................(4.63)}$$

The temperature T and angular velocity Ω of the BTZ blackhole are given by,

$$T = \frac{\delta M}{\delta S_{BH}} = 2S_{BH} - \frac{J^2}{2S_{BH}^3} \quad \text{..........................(4.64)}$$

and $$\Omega = \frac{\delta M}{\delta J} = \frac{J}{2S_{BH}^2} \quad \text{..........................(4.65)}$$

The heat capacity of the BTZ blackhole is given by,

$$C = \frac{T}{\frac{\delta^2 M}{\delta S_{BH}^2}} = \frac{S_{BH}(4S_{BH}^4 - J^2)}{4S_{BH}^4 + 3J^2} \quad \text{......................(4.66)}$$

It is notable that the heat capacity of the BTZ blackhole is a positive quantity when $4S_{BH}^4 > J^2$. It reduces to zero when $4S_{BH}^4 = J^2$ and negative for $4S_{BH}^4 < J^2$.

CHAPTER FIVE

BLACKHOLE RADIATION AND UNCHARGED PARTICLE TUNNELING

5.1 Background history of Hawking radiation:

Here we would like to give a historical background of Hawking radiation following the reference [84]. According to this reference the precursor of Hawking's discovery of emission by blackholes were the calculation by Parker[130,131,132,133] and Fulling[134,135]of particle creation by expanding universes, which developed the concepts of Bogoliubov transformation[136] in time dependent geometries that were later used by Hawking. It was a surprise to everyone, including Hawking , that the emission from a blackhole persisted even when the blackhole become effectively static. The first prediction of emission by a blackhole was made by Zel'dovich[137,138] who pointed out on heuristic grounds that a rotating blackhole should amplify certain waves and there should be an analogous quantum effect of spontaneous radiation of energy and angular momentum. Later Misner [139] and Starobinski 140] confirmed the amplification by a Kerr blackhole of scalar waves in the 'supperradiant regime' and Bekenstein[141] showed that amplification should occur for all kinds of waves with positive energy density. The quantum effect predicted by Zel'dovich was rediscovered by Larry Ford and Page .

The argument for this spontaneous radiation was that in a quantum analysis the amplification of waves is stimulated emission of quanta, so that even in absence of incoming quanta one should get spontaneous emission. By using the relation between the Einstein coefficients for spontaneous and stimulated emission , one can calculate the spontaneous rate from the amplification factor, as Starobinski [140] noted, at least when the spontaneous emission probability is much less than unity.

A problem arose for neutrinos in that Unruh[142] showed that their waves are never amplified. This result violated Bekenstein's conclution and seemed to be a breakdown of Hawking area theorem[8]. The reason for the violation was traced to a negative local energy density of the classical neutrino waves at the horizon. However Feynman suggested that the lack of amplification might be due to the Pauli exclusion principle, so that incident neutrinos suppress spontaneous emission which otherwise occurs. The amplification factor would then less than unity, since the calculation of an unquantized neutrino wave cannot directly show the spontaneous emission but only how the emission changes as the incident flux is varied.

One might be surprised to find such a difference between integral and half-integrall spins showing up in the behavior of their unquantized waves, but this is merely an illustration of the connection between spin and statistics. Pauli [143] has shown that the half-integral spins must be assigned anticommutation relations in order to get a positive energy density, which is precisely what the unquantized nutrino waves violate in not showing supperradiance.

Unruh [144] made a formal calculation of second quantization of scalar and Neutrino fields in the complete Kerr blackhole. Ford[145] quantized the massive scalar field in a somewhat different way with similar results. However , Unruh noted that the actual situation might be different , with no past horizon but the blackhole formed by collapse. Nevertheless , neither he nor any of the discoverers of the spontaneous emission attempted to calculate that situation.

In 1973, Hawking at Cambridge University heard of this work through Douglas Eardly and D.N. Page and so while his visit in Moscow, where Soviet scientists Zel'dovich and Starobinski showed him that according to the quantum uncertainty principle , rotating blackholes should create and emit particles. Believing in the reality of the spontaneous emission but wishing to put its derivation on a firmer footing , Hawking dared to attempt the difficult calculation of field theory during the collapse and formation of a blackhole. Separating out the essential elements , Hawking found how to calculate the particle emission at late times, after the collapse had settled down to form a stationary blackhole. At first Hawking got an infinite number of particles emitted, but then he discovered that the infinity corresponded to emission at a steady rate. However , the emission was not only in the supperradiant states or modes but in all modes that could come from the blackhole.

Hawking initially did not believe this result. Thinking that the emission might be an artifact of the spherical symmetry he had assumed, Hawking considered nonspherical collapse and got the same emission. Then he tried to putting in a cutoff on the frequencies of the modes in the initial states before the collapse, but that eliminated all the emission , including the spontaneous emission in the supperradiant modes that Hawking was certain existed. Perhaps most convincing to Hawking was the fact that the emission rate was just that of a thermal body with the same absorption probabilities as the blackhole and with a temperature (in geometrical units) equal to the surface gravity of the hole divided by 2π. This result holds for fields of any spin and seemed to confirm some thermodynamic ideas of Bekenstein[3]. However , before the emission process was discovered, Bardeen, Carter, and Hawking [1] had argued against Bekenstein's suggestion of a blackhole temperature proportional to surface gravity. Thus Bekenstein's ideas were originally not a motivation for Hawking's calculation.

As word of his calculation began to spread, Hawking published a simplified version of it in Nature[9]. However, even at this stage Hawking was not certain of the result and so expressed the title as a question , "Blackhole explosions?" He noted that the calculation ignored the change in the metric due to the particles created and to quantum fluctuations. One objection raised by several people was that the calculation seemed to give a very high energy flux just outside the horizon, which might prevent the blackhole from forming at all. Hawking later answered this and other problems by a more detailed version of the calculation [4],which shows that an in falling observer would not see many particles near the horizon. However, it might be noted that there is still some controversy about the existence of particles there. The back reaction of the particles created would , in Hawking's view , simply be to reduce the mass of the hole by the amount of the energy radiated away.

Presumably quantum fluctuation of the metric itself can give rise to the emission of gravitons in addition to the emission of other particles calculated as if the geometry were fixed. By considering linearized fluctuations in the metric about a given background , the emission of gravitons can be handled in the same manner as the emission of any other particles , though one might argue that gravitons emission depends more fundamentally upon the assumed fluctuations in the metric. Therefore, an observed consequences of graviton emission can be viewed as testing whether gravity is quantized.

Hawking has argued that quantum mechanics allows small deviations of the action from the extreme value that gives the classical field equations for matter and geometry. Thus the classical equations can be violated in the small region near a blackhole, giving rise to the emission of matter or gravitational waves, but the equations cannot be violated significantly on a very large surface surrounding the hole. Therefore, quantities determined by surface fluxes at infinity do remain conserved; energy, momentum, angular momentum, and charge. This is the basis for arguing that the emission carries away the quantities of the hole which otherwise would be constant. Note that baryon and lepton numbers are not observed to be connected with long-range fields, so they presumably cannot be determined by surface fluxes at infinity and thus would not be conserved globally by the blackhole emission process.

The thermal emission first calculated by Hawking has been verified by several subsequent calculations. Boulware[146] and Davies [147] have calculated the emission from collapsing shell. Gerlach [148] has interpreted the emission as parametric amplification of the zero-point oscillations of the field inside the collapsing object. DeWitt[149] has given detailed derivations of both the spontaneous emission process in the complete Kerr metric(with no particles coming out of the past horizon) and of the thermal emission from a blackhole formed by collapse. Unruh [150] has found that his derivation in the complete Kerr metric will give not only the spontaneous but also the thermal emission if the boundary condition at the past horizon is changed from no particles seen by an observer at fixed radius just outside the horizon to no particles seen by an observer freely falling along the horizon. Wald [151], Parker[152] and Hawking[153]have calculated the density matrix of the emitted particles and find that it , as well as the expected number in each mode, is precisely thermal. Bekenstein [154] has given an information theory argument of why this should be so. Hartle and Hawking[12] have done a path-integral calculation of the probability for a particle to propagate out of a blackhole from the future singularity and show that this method also leads to the same thermal spectrum .In summary , the thermal emission from a blackhole has been derived in a variety of way by several people, so its predictions seems to be a clear consequence of our present theories of quantum mechanics and general relativity.

5.2 Nature of Hawking radiation:

The spacetime associated to gravitational collapse to a blackhole cannot be everywhere stationary, so we expect particle creation. But the exterior spacetime is stationary at late times, so we might expect particle creation to be just a transient phenomenon determined by details of the collapse. But the infinite time dilation at the horizon of a blackhole means that particles created in the collapse take arbitrarily long to escape – suggests a possible flux of particles at late times that is due to the existence of the horizon and independent of the details of the collapse. There is such a particle flux , and it turns out to be thermal- this is known as Hawking radiation or blackhole radiation.

Hawking's discovery is one of the most well established predictions obtained from the study of quantum mechanics linked to Einstein's general relativity. According to Hawking's the radiation from a blackhole is essentially thermal: the blackhole emits field quanta of all frequencies , according to the usual black body spectrum, corrected with a factor that accounts for the scattering from the spacetime curvature[155]. In particular , for a Schwarzschild blackhole with mass M the characteristic temperature is given by

$$T = \frac{\hbar c^3}{8\pi G k_B M} \quad \text{.................................(5.2.1)}$$

where k_B is the Boltzmann factor.

Some authors have raised the possibility that the Hawking radiation might in fact have a discrete spectrum. The idea that blackhole may have a discrete spectrum was first proposed by Bekenstein in 1974[156] who used an analogy between the horizon area A for Kerr blackhole proportional to the squared irreducible mass [157]

$$M_{irr}^2 = \frac{A}{16\pi} \quad \text{..............................(5.2.2)}$$

and an action integral $\oint p dq$ of a periodic mechanical system. This analogy was based on a fact that the irreducible mass behaves as an adiabatic invariant i.e. remain unchanged in reversible processes. Using this analogy and Bohr-Sommerfield quantization condition Bekenstein obtained the discrete spectrum[157]

$$M_{irr}^2 \sim M_P^2 n \quad \text{...............................(5.2.3)}$$

where M_P is the Planck mass and n is an integer. Later in 1986 V. Mukhanov[158] and Kogan[159] independently revived this idea using completely different arguments. This problem attract more attention and has been discussed in several interesting papers[160,161,162,163,164] where the discrete spectrum was derived using new ideas. We also note that quantization of the area operator in quantum gravity has been obtained in [165,166] using the loop representation.

However , we recall the Bekenstein proposal[156] which can be described by the eigenvalues of the blackhole event horizon area of the form

$$A_n = \alpha n l_{p_l}^2 \quad \text{.....................(5.2.4)}$$

where α is the constant of order one , n ranges over positive integer and l_{p_l} is given by

$$l_{p_l} = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} \quad \text{...............................(5.2.5)}$$

is the Planck length.

In particular case of a Schwarzschild blackhole the fact that equation(4.2.4) implies a discrete spectrum for Hawking radiation can be seen by recalling that the area of such a blackhole with mass M is

$$A = \frac{16\pi G^2 M^2}{c^4} \quad \text{.............................(5.2.6)}$$

and so it follows from equation (5.2.4) and (5.2.6) that the angular frequencies of the quanta of Hawking radiation are integer multiples of the fundamental angular frequency[155]

$$\omega_0 = \alpha \frac{c^3}{32\pi GM} \quad \text{...................................(5.2.7)}$$

For example, if M is ten solar masses or $2\times10^{31}\,kg$, then ω_0 is of order of $0.1kHz$, which is roughly the resolving power of an ordinary portable radio receiver[155].

Indeed we have two theories about the explanation of Hawking radiation which are contradicting experimental predictions. According to Hawking's opinion a continuous blackbody spectrum for blackhole radiation, whereas Bekenstein's proposal states that the spectrum is discrete. The qualitative difference is that; one easily sees from Wien's displacement law that the fundamental angular frequency ω_0 of equation (5.2.7) is near the angular frequency corresponding to the maximum of the blackbody spectrum with the temperature given by equation(5.2.1).

Now, one may asked, is there any relationship between two theories? We shall, for the sake of convenience , refer to the theories based on Hawking's result and Bekenstein's proposal, respectively, by continuous and discrete theories of blackhole radiation.

The starting points of the two theories are completely different. The continuous theory developed based on the semi-classical theory where spacetime is treated classically and matter fields are treated as quantum mechanically. On the other hand , Bekenstein's proposal and hence discrete theory are supposed to be quantum theories of vacuum blackhole spacetimes. If one adopts the viewpoint that Bekenstein's proposal arises from a quantum theory of vacuum spacetimes, one may feel justified to regard the discrete theory as more fundamental than the continuous one. More precisely, one may expect the continuous theory to emerge as the semi-classical limit of the discrete theory when the effects of matter fields are taken into account. This implies that if the effect of quantized matter fields are assumed to overshadow the effect of quantum mechanics of the spacetime, the discrete theory should reduce to the continuous one.

In addition, the blackhole can be formed either due to the bound motion of matter or due to its unbound motion. In quantum mechanics we are used to the fact that we have a discrete mass spectrum for a bound motion and a continuous spectrum for an unbound motion. One of the possibilities is that the blackhole total mass has a discrete spectrum or its spectrum is continuous. If it is continuous , then it is possible for both bound and unbound motion. If it is discrete , then the corresponding discrete quantum number has the same origin both for bound and unbound motion[167]. But the bound motion has also a "conventional" quantum number(or numbers). And the unbound motion has also a "conventional" continuous parameter(or parameters). From this it follows that if the mass of the blackhole is quantized i.e. discrete, the quantum blackhole state is described not only by its mass but also by some other parameter(s), discrete or continuous.

Indeed , the spectrum of radiation coming from a blackhole should have a very complicated structure. According to the Bohr postulate, the spectrum of emitted quanta must correspond to energy level spacing of quantum system. If the blackhole spectrum depends on extra parameters , the radiation spectrum would have a fine structure in the case of discrete additional parameters or it is continuous in the case of a continuous additional parameters. So we conclude that the radiation spectrum "remembers" the way how the blackhole is formed. The spectrum seems to be not the universal one, but in accordance with the well known semi-classical result by Hawking the spectrum of radiation from a blackhole is universal-it is thermal.

5.3 Blackbody radiation vs blackhole radiation:

we know that a blackbody is an ideal body which allows the whole of the incident radiation to pass into itself(without reflecting energy) and absorbs within itself this whole incident radiation(without passing on the energy).This property is valid for radiation corresponding to all wavelengths and to all angles of incidence. Therefore , the blackbody is an ideal absorber of incident radiation. But a blackbody not only absorbs radiation ideally. If we consider a blackbody at constant temperature placed inside a fully insulated cavity of arbitrary shape, whose walls are also formed by ideal blackbodies at constant temperature, which initially differs from the temperature of the body inside. After some time the blackbody and the closed cavity will have a common equilibrium temperature. Under equilibrium conditions the blackbody must emit exactly the same amount of radiation as it absorbs. This phenomenon is known as blackbody radiation.

On the other hand , in classical mechanics a blackhole absorbs all incident radiation or matter falling on it , so a blackhole can be treated as a blackbody. But we have seen that nothing from inside can get out of a blackhole, so it would appear that it cannot be a source of radiation. In this sense , blackhole don't seem to fit comfortably into thermal physics.

However, blackbody radiation is a quantum phenomenon. Planck invented his constant in order to describe it. On the other hand British physicist Hawking studied the quantum theory of electromagnetism near the blackholes, he found that blackholes actually emit radiation, that in fact has a blackbody spectrum. It should be no surprise that the blackhole radiation phenomenon lies in quantum uncertainty. All over spacetimes,the quantum electromagnetic field is undergoing the little negative-energy quantum fluctuations. Normally they are harmless and invisible , because the negative-energy photons disappear as quickly as they form. But near the horizon of a blackhole , it is possible for such a photon to form outside the hole and crosses into it.

Once inside, it is actually viable as we remarked earlier, it is possible to find trajectories for photons inside the horizon that have negative total energy. So such a photon can just stay inside, and that leaves its positive energy partner outside on its own. It has no choice but to continue moving outwards. It becomes one of the photons of the Hawking radiation.

In this picture nothing actually crosses the horizon from inside to out. Instead the negative energy photon falls in, freeing the positive energy photon. The net result of this is that the hole loses mass: the negative energy photon makes a negative contribution to the mass of the hole when it goes in.

Once we accept that blackholes can radiate, then it is possible to estimate the wavelength of the radiation that blackholes emit. If we take a photon of wavelength λ equal to the radius of the blackhole then the energy is equal to

$$E = h\nu = \frac{hc}{\lambda} = \frac{hc^3}{2GM} \qquad \text{..........................(5.3.1)}$$

(for Schwarzschild blackhole)

If we consider the blackholes are indeed blackbodies, absorbing everything that falls on them and emitting light then their temperature T should be at least approximately related to this energy by setting $E = kT$, leading to the following estimate of the temperature of a blackhole

$$T = \frac{hc^3}{2kGM} \qquad \text{..........................(5.3.2)}$$

Without any reason we take the wavelength equal to the radius of the hole rather than , say, its diameter or circumference and we must expect that the details of quantum theory and spacetime curvature will not be encapsulated in such a simple dimensional argument. Nevertheless we see that only a factor larger than the one Hawking found, which is called the Hawking temperature T_H is

$$T_H = \frac{hc^3}{16\pi^2 kGM} \approx 6\times10^{-8}\left(\frac{M}{M_\otimes}\right)^{-1} K \qquad \text{........................(5.3.3)}$$

Obviously, this is so small for stellarmass and supermassive blackhole that it has little relevance to astrophysics. But Hawking's discovery is widely regarded as one of the first real steps toward a quantum theory of gravity. Although we have no such theory , many physicist expect that it must predict the Hawking radiation.

5.4 Luminosity and life time of blackholes:

The luminosity is the rate of energy of all types is radiated by an object in all directions. The luminosity of a star depends on its size and its temperature, varying as the square of the radius and the fourth power of the absolute surface temperature. Our sun is a medium sized star with a luminosity of 3.8×10^{33} ergs per sec. The luminosities of other stars are commonly expressed in terms of the sun's luminosity. The known luminosities of stable stars range from about 10^{-6} that of the sun for a relatively cool white dwarf to about 10^{6} times that of the sun for the hottest supergiant star.

If we assumed that the blackhole are completely blackbody, then we can apply the Stefan-Boltzmann law of blackbody radiation. According to this law , the luminosity of a blackhole is given by

$$L_{bh} = \sigma A T^4 \quad \ldots\ldots\ldots\ldots(5.4.1)$$

where σ is the Stefan-Boltzmann constant given by

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} \quad \ldots\ldots\ldots\ldots(5.4.2)$$

A is the area of the surface that radiates. In this case the surface area is the surface area of the blackhole, T is the temperature of the blackhole. Applying the formula equation (5.4.1) and putting the values of A and T for different types of blackhole we can obtain the luminosity of the different types of blackhole. Here we discuss only Schwarzschild blackhole.

The relevant surface for the Hawking radiation in the surface area of the a sphere with radius of the Schwarzschild blackhole radius, because that's where the radiation originates; so

$$A = 4\pi R^2_{Sch.} = 16\pi\frac{G^2M^2}{c^4} \quad \ldots\ldots\ldots\ldots(5.4.3)$$

Substituting this in equation (5.4.1) we have the luminosity for the Schwarzschild blackhole as

$$L_{Sch.} = \frac{2\pi^5 k^4}{15h^3c^2}\cdot\frac{16\pi G^2M^2}{c^4}T^4 \quad \ldots\ldots\ldots\ldots(5.4.4)$$

Using temperature T given by equation (5.2.1)

$$T = \frac{\hbar c^3}{8\pi GkM} = \frac{hc^3}{16\pi^2 kGM} \quad \ldots\ldots\ldots\ldots(5.4.5)$$

From equation (5.4.4) we have the luminosity of Schwarzschild blackhole as

$$L_{Sch.} = \frac{32\pi^6 k^4 G^2 M^2}{15h^3c^6}\left[\frac{hc^3}{16\pi^2 kGM}\right]^4 = \frac{1}{15360\pi}\cdot\frac{\left(\frac{c\hbar}{G}\right)^2}{M^2}\cdot\frac{c^5}{G} \quad \ldots\ldots\ldots\ldots(5.4.6)$$

In equation (5.4.6) we see that the first factor is , of course , just a pure number. The numerator of second factor is the quantity $\frac{\hbar c}{G}$ which is the square of the Planck mass M_P i. e. $M_P^2 = \frac{\hbar c}{G}$. So the second factor is dimensionless, being the ratio of the square of two masses. The third factor is known as Einstein luminosity.

The Einstein luminosity is large, but the blackhole only approaches this luminosity when its mass as small as the Planck mass. For an ordinary hole, the factor $\frac{1}{M^2}$ reduces the luminosity drastically. For example a $10M_{\otimes}$ blackhole radiates $10^{-30}W$!

Now we turn to discuss the lifetime of a blackhole. Through the Hawking radiation , blackholes gradually lose mass. Unlike other bodies , the smaller they get, the higher their temperature goes according to equation (5.4.5), so the loss mass accelerates. In the picture of Hawking radiation , the escaping member of a virtual particle pair carried away energy from blackhole and so the blackhole losses mass as a result. Eventually the blackhole losses all its energy , or equivalent mass, and evaporates and this issue that blackhole evaporation is one of the most surprising discoveries of the past 70's decades. Now we would like to derive a formula to calculate the lifetime of blackhole.

The power of Hawking radiation is just the same as the luminosity of a blackhole

$$P = L = \frac{hc^6}{30720\pi^2 G^2 M^2} \quad \text{..................................(5.4.7)}$$

On the other hand the amount of energy carried away by Hawking radiation is the same as the energy loss by the blackhole. Therefore the power of Hawking radiation is the rate of loss of total energy E of a blackhole i.e.

$$P = -\frac{dE}{dt} \quad \text{...................................(5.4.8)}$$

From Einstein mass energy relation $E = Mc^2$, we have

$$P = -\frac{dE}{dt} = -c^2 \frac{dM}{dt} \quad \text{.............................(5.4.9)}$$

Comparing equation (5.4.7) and (5.4.9) we get

$$\frac{hc^6}{30720\pi^2G^2M^2} = -c^2\frac{dM}{dt}$$

$$or, -Kdt = M^2dM \qquad \text{...............................(5.4.10)}$$

where $K = \frac{hc^4}{30720\pi^2G^2}$.

As a blackhole slowly evaporates, its mass drops from M_0 (its initial mass) to zero. The time required for evaporation starts from zero to t (total time for evaporation). So integrating equation (5.4.10) we have

$$-K\int_0^t dt = \int_{M_0}^0 M^2\,dM \qquad \text{........................(5.4.11)}$$

Finishing the integral we obtain

$$t = \frac{M_0^3}{3K} \qquad \text{...............................(5.4.12)}$$

Therefore the blackhole lifetime formula can be written as

$$t_{lifetime} = \frac{M_0^3}{3K} \qquad \text{...............................(5.4.13)}$$

The equation (5.4.13) tells us that the lifetime of a blackhole is proportional to the cube of its mass. This implies that a massive blackhole takes proportionally much longer time to evaporate and the process of evaporation accelerates as the blackhole slowly loses its mass. Moreover , if we look the temperature formula for blackhole , $T = \frac{hc^3}{16\pi^2kGM}$, then we see that as the blackhole loses its mass , its temperature increases. When a blackhole get very small, its temperature may become so high that it may burn up and cause an explosion. For example, the lifetime of a blackhoole having the mass of our sun is

$$t_{lifetime} = \frac{M_{sun}^3}{3K} \approx \frac{(1.99\times10^{30}\,kg)^3}{3\times3.98\times10^{15}\,kg^3g^{-1}} \approx 2\times10^{67}\,years. \qquad \text{...............(5.4.14)}$$

The lifetime of such a blackhole is even longer than that of our universe. If we set $t_{lifetime}$ to the present age of the universe, we obtain a minimum mass such a 'primordial' blackhole must have had (assuming it Hawking radiates) to survive to the present day. This mass is approximately $\sim10^{15}\,g$ [168].

Now we discuss the hawking radiation as tunneling process of massless particle from the event horizon of different types of blackholes as follows:

5.5 Uncharged particle tunneling from Schwarzschild blackhole:

5.5.1 Painleve- like coordinate transformation and null geodesics of Schwarzschild blackhole :

The line element of the Schwarzschild blackhole is given by,

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -f(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \quad \text{.................................(5.5.1)}$$

where $f(r) = g(r) = 1 - \frac{2M}{r}$.

The study of this geometry over the course of several decades revealed a number of remarkable surprises. First , the existence and physical relevance of pure vacuum 'blackhole' solutions; second, the incompleteness of the spacetime covered by the original Schwarzschild coordinates and the highly non-trivial global structure of its completion; third, the dynamic nature of the physics in this geometry despite its static mathematical form, revealed perhaps most dramatically by the Hawking radiation[4].

Since the metric (5.5.1) has the coordinate singularity at the horizon, so we have need to a coordinate transformation such that there is no singularity at the horizon. For this reason we should adopt Painleve coordinate transformation [173] to investigate the Hawking radiation as tunneling process. This is easily accomplished via transformation,

$$t_s = t - \int\sqrt{\frac{1-g(r)}{f(r)g(r)}}dr \quad \text{..........................(5.5.2)}$$

This transformation reduces the metric (5.5.1) in the form,

$$ds^2 = -(1-\frac{2M}{r})dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

$$\text{................(5.5.3)}$$

Now there is no singularity at the horizon and the true character of the spacetime is stationary but not static. Thus it is possible to define an effective "vacuum" state of a quantum field by requiring that it annihilate modes which carry negative frequency with respect to t ; such a state will look essentially empty to a freely-falling observer as he or she passes through the horizon.

The radial null geodesics are given by putting $ds^2 = d\theta = d\phi = 0$ in (5.5.3)

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r}} \quad \text{.....................................(5.5.4)}$$

where the positive sign (+) represents the outgoing geodesics and the negative sign (-) represents the ingoing geodesics under the implicit assumption that t increases towards the future.

5.5.2 Tunneling rate of Schwarzschild blackhole:

We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass of the blackhole. We consider the particle as an ellipsoid shell of energy ω. When the particle's self-gravitation is taken into account, then equation (5.5.3) and (5.5.4) should be modified. Krause and Wilczek [13] found that , when the blackhole mass is held fixed and the total ADM (Arnowitt-Deser-Misner) mass allowed to vary , a shell of energy ω moves in the geodesics of a spacetime with M replaced by $M+\omega$. If instead we fix the total mass and allow the hole mass to fluctuate, then the shell of energy ω travels on the geodesics given by the line element,

$$ds^2 = -\left\{1 - \frac{2(M-\omega)}{r}\right\}dt^2 + 2\sqrt{\frac{2(M-\omega)}{r}}dtdr + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \qquad \text{.......................................(5.5.5)}$$

and the shell of energy ω will move along the modified null geodesic in the radial direction ,

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M-\omega)}{r}} \qquad \text{.....................................(5.5.6)}$$

Since the typical wavelength of the radiation is of the order of the size of the blackhole , so one might doubt whether a point particle description is appropriate. However , when the outgoing wave is traced back towards the horizon, its wavelength , as measured by local fiducial observers, is ever-increasingly blue-shifted. Near the horizon, the radial wave number approaches infinity and the point particle, or WKB, approximation is justified.

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} is expressed as,

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} P_r \, dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{P_r} dP_r' \, dr \qquad \text{.....................................(5.5.7)}$$

where r_{in} and r_{out} are the respective initial and final radii of the blackhole , P_r is the canonical momentum conjugate to r. The trajectory between these two radii is the barrier that the particle must tunnel through.

In order to evaluate the integral, we employ the Hamilton's equation $\dot{r} = \left.\frac{dH}{dP_r}\right|_r$, change variable from momentum to energy, we obtain

$$\operatorname{Im} S = \operatorname{Im} \int_{M_{in}}^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}} dH = \operatorname{Im} \int_{M}^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{1-\sqrt{\frac{2(M-\omega')}{r}}} (-d\omega') \quad \text{.......................(5.5.8)}$$

where $dH = -d\omega'$ because total energy $H = M - \omega'$ with M constant.

The integral can be done by deforming the contour, so as to ensure that the positive energy solutions decay in time. Doing the ω' integral first we obtain,

$$\operatorname{Im} S = -\pi \int_{r_{in}}^{r_{out}} r\, dr = \frac{\pi}{2}\left[r_{in}^2 - r_{out}^2\right] = \pi(4M\omega - 2\omega^2).$$

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{-2\pi(4M\omega - 2\omega^2)} \quad \text{..........................(5.5.9)}$$

Using Bekenstein-Hawking entropy formula we have,

$$S_{BH}(M) = \frac{A}{4} = 4\pi M^2$$

$$S_{BH}(M-\omega) = 4\pi(M-\omega)^2.$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M) = -2\pi(4M\omega - 2\omega^2) \quad \text{.......................(5.5.10)}$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission.

Substituting equ.(5.5.10) into equ.(5.5.9) we have,

$$\Gamma \sim e^{\Delta S_{BH}} \quad \text{..(5.5.11)}$$

From equ.(5.5.9) we see that when the quadratic term in ω neglected, then it reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature i.e. $e^{-\beta\omega}$ where $\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = 8\pi M$. The second term represents correction arises from the physics of energy conservation, which self-consistently raises the effective temperature of the hole as it radiates. Also from equ.(5.5.10) we see that the radiation spectrum is not purely thermal.

5.6 Uncharged particle tunneling from R-N blackhole:

5.6.1 Painleve- like coordinate transformation and null geodesics of R-N blackhole:

The line element of Reissner-Nordström blackhole is given by,

$$ds^2 = -(1-\frac{2M}{r}+\frac{Q^2}{r^2})dt^2 + (1-\frac{2M}{r}+\frac{Q^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{.................(5.6.1)}$$

To investigate the Hawking radiation as tunneling process we should adopt Painleve coordinate transformation. According to the transformation equation (5.5.2) the metric (5.6.1) reduces to,

$$ds^2 = -f(r)dt^2 + 2\sqrt{f(r)}\sqrt{\frac{1}{g(r)}-1}\, dtdr + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \quad \text{.............(5.6.2)}$$

where $f(r) = g(r) = 1-\frac{2M}{r}+\frac{Q^2}{r^2}$.

The metric(5.6.2) can be rewritten as,

$$ds^2 = -(1-\frac{2M}{r}+\frac{Q^2}{r^2})dt^2 + 2\sqrt{\frac{2M}{r}-\frac{Q^2}{r^2}}\, dtdr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{..........(5.6.3)}$$

The radial null geodesics are given by,

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r}-\frac{Q^2}{r^2}} \quad \text{...(5.6.4)}$$

where the positive sign (+) represents the outgoing geodesics and the negative sign (-) represents the ingoing geodesics under the implicit assumption that t increases towards the future.

5.6.2 Tunneling rate of R-N blackhole:

We have already given the tunneling process of Hawking radiation for the Schwartzschild blackhole. With the same sprit we modified the equation (5.6.3) as,

$$ds^2 = -(1-\frac{2(M-\omega)}{r}+\frac{Q^2}{r^2})dt^2 + 2\sqrt{\frac{2(M-\omega)}{r}-\frac{Q^2}{r^2}}\, dtdr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{..........................(5.6.5)}$$

and the shell of energy ω will move along the modified null geodesic in the radial direction ,

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M-\omega)}{r} - \frac{Q^2}{r^2}} \quad \text{.....................................(5.6.6)}$$

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} is expressed as,

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} P_r \, dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{P_r} dP_r' \, dr \quad \text{....................................(5.6.7)}$$

where r_{in} and r_{out} are the respective initial and final radii of the blackhole , P_r is the canonical momentum conjugate to r. The trajectory between these two radii is the barrier that the particle must tunnel through.

We employ the Hamilton's equation $\dot{r} = \left.\frac{dH}{dP_r}\right|_r$, to evaluate the integral. By changing variable from momentum to energy, we obtain

$$\operatorname{Im} S = \operatorname{Im} \int_{M_{in}}^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}} dH = \operatorname{Im} \int_{M}^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r} - \frac{Q^2}{r^2}}} (-d\omega') \quad \text{.................(5.6.8)}$$

where $dH = -d\omega'$ because total energy $H = M - \omega'$ with M constant.
The integral can be done by deforming the contour, so as to ensure that the positive energy solutions decay in time. Doing the ω' integral first we obtain,

$$\operatorname{Im} S = -\pi \int_{r_{in}}^{r_{out}} r \, dr = \frac{\pi}{2}\left[r_{in}^2 - r_{out}^2\right]$$

$$= -\pi\left[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - Q^2} - M^2 - M\sqrt{M^2 - Q^2}\right]$$

The tunneling rate is therefore,

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{2\pi\left[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - Q^2} - M^2 - M\sqrt{M^2 - Q^2}\right]}$$

$$\text{..........................(5.6.9)}$$

Using Bekenstein-Hawking entropy formula we have,

$$S_{BH}(M) = \frac{A}{4} = \pi\left[2M^2 + 2M\sqrt{M^2 - Q^2} - Q^2\right]$$

$$S_{BH}(M-\omega) = \pi\left[2(M-\omega)^2 + 2(M-\omega)\sqrt{(M-\omega)^2 - Q^2} - Q^2\right]$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$$

$$= 2\pi\left[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - Q^2} - M^2 - M\sqrt{M^2 - Q^2}\right]$$

$$\text{..............................(5.6.10)}$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. Substituting equ.(5.6.10) into equ.(5.6.9) we have,

$\Gamma \sim e^{\Delta S_{BH}}$ ....(5.6.11)

From equ.(5.6.9) we see that when the quadratic term neglected, then it reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature i.e. $e^{-\beta\omega}$ where $\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = \frac{2\pi\left[M + \sqrt{M^2 - Q^2}\right]^2}{\sqrt{M^2 - Q^2}}$. The other terms represents correction arises from the physics of energy conservation, which self-consistently raises the effective temperature of the hole as it radiates. Also from equ.(5.6.11) we see that the radiation spectrum is not purely thermal.

5.7 Uncharged particle tunneling from Kerr Blackhole:

5.7.1 Dragged Painleve-Gullstrand Kerr metric and null geodesics of Kerr Blackhole:

The line element of Kerr metric in the Boyer-Lindquist coordinate system is given by,

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a\,dt]^2$$

..............................(5.7.1)

where

$$\Delta = r^2 - 2Mr + a^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum and r is the radial distance from the center of the body. The equation of the event horizon is given by,

$\Delta = 0$ which gives, $r_\pm = M \pm \sqrt{M^2 - a^2}$, $M^2 > a^2$.

In order to investigate Hawking radiation as tunneling from Kerr blackhole we first adopt dragging coordinate system to overcome two difficulties. First , the event horizon $r_\pm = M \pm \sqrt{M^2 - a^2}$ does not coincide with the infinite red-shift surface $r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$, which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Second , as there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (5.7.1) into a dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ ....(5.7.2)

where Ω is the angular velocity.

For the metric (5.7.1) we have,

$$g_{00} = \frac{-(\Delta - a^2 \sin^2\theta)}{\rho^2},\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \frac{\sin^2\theta[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta]}{\rho^2}$$

$$g_{03} = \frac{-a\sin^2\theta[(r^2+a^2)-\Delta]}{\rho^2}$$

From (5.7.2), $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}} = \frac{a[(r^2+a^2)-\Delta]}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$

....(5.7.3)

At the horizon the angular velocity becomes,

$\Omega_h = \frac{a}{r_+^2 + a^2}$ ....(5.7.4)

The line element (5.7.1) in the dragging coordinate system becomes,

$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dr^2 + g_{22} d\theta^2$ ....(5.7.5)

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2}{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}$.

The line element (5.7.5) represents a 3-dimensional hypersurface of 4-dimensional spacetime. Although in the dragging coordinate system does not has any singularity at the event horizon, and we can also get the Hawking pure thermal spectrum , yet this coordinate system is not what we want because it is not flat Euclidean space in radial to constant time slices. So we continue performing a general Painleve-like coordinate transformation[173]. For this transformation we set,

$dt_K = dt + F(r,\theta)dr + G(r,\theta)d\theta$ ....(5.7.6)

where $F(r,\theta)$ and $G(r,\theta)$ are two determined functions of r and θ , and satisfy the integrability condition ,

$\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$ ....(5.7.7)

Thus from (5.7.5) we obtain,

$$ds^2 = \hat{g}_{00}\, dt^2 + \left\{ \hat{g}_{00}\, F^2(r,\theta) + g_{11} \right\} dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2G(r,\theta)\, \hat{g}_{00}\, dtd\theta$$

$$+ 2F(r,\theta)G(r,\theta)\, \hat{g}_{00}\, drd\theta + 2F(r,\theta)\, \hat{g}_{00}\, dtdr$$

................................(5.7.8)

We demand that constant time- slices are flat Euclidean space in radial. So we set,

$$\hat{g}_{00}\, F^2(r,\theta) + g_{11} = 1$$

$$\Rightarrow F(r,\theta) = \pm \sqrt{\frac{1-g_{11}}{\hat{g}_{00}}} \qquad \text{..............................(5.7.9)}$$

From equation (5.7.7), $G(r,\theta) = \int \frac{\delta F(r,\theta)}{\delta\theta} dr + C(\theta)$ ....(5.7.10)

where $C(\theta)$ is an arbitrary analytic function of θ .

Substituting the value of $F(r,\theta)$ into equation (5.7.8) we get,

$$ds^2 = \hat{g}_{00}\, dt^2 + dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2\sqrt{\hat{g}_{00}(1-g_{11})}\; G(r,\theta) drd\theta$$

$$+ 2\, \hat{g}_{00}\, G(r,\theta)\, dtd\theta \pm 2\sqrt{\hat{g}_{00}(1-g_{11})}\; dtdr$$

...................(5.7.11)

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon.

According to Landau's theory of the coordinate clock synchronization[174] in a spacetime decomposed in 3+1 dimension, the difference of coordinate times of two events taking place simultaneously in different place is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}} dx^i \qquad (i = 1,2,3) \qquad \text{........................(5.7.12)}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy

$$\frac{\delta}{\delta x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\delta}{\delta x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad , \ (i, j = 1,2,3) \qquad \text{......................(5.7.13)}$$

Now the metric (5.7.11) in the new coordinate system , which we called the dragged Painleve-Gullstrand Kerr metric, has a number of attractive features : (1) the metric is well-behaved at the event horizon; (2) the time coordinate t represents the local proper time for radially free-falling observers; (3) the hypersurfaces of constant time-slices are just flat Euclidean space in the oblate spheroidal coordinates; (4) by substituting the components of the metric (5.7.11) into equation (5.7.13), we see that the metric satisfy the Landau's condition of the coordinate clock synchronization $\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$; (5) the infinite red-shift surface coincide with the event horizon surface so that the WKB approximation can be used. These attractive features are very advantageous for us to discuss Hawking radiation as tunneling and to do an explicit computation of the tunneling probability at the event horizon.

Now in order to investigate the tunneling process we evaluated the radial null geodesics described by equation (5.7.11) . Since the tunneling processes take place near the event horizon, so we may consider a particle tunneling from the event horizon as an ellipsoid shell . To conserve the symmetry of the spacetime , we think the particle should be still an ellipsoid shell during the tunneling process i.e. the particle does not have motion in θ-direction[179]. Under these condition we obtain the radial null geodesics from equation (5.7.11) $(ds^2 = 0 = d\theta^2)$ as

$$\dot{r} = \frac{\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta)}}{\sqrt{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta}} \qquad \text{.......................(5.7.14)}$$

(where a dot denotes differentiation with respect to t and the positive sign $(+)$ represents an outgoing geodesics and the negative sign $(-)$ represents an ingoing geodesics.

5.7.2 Tunneling rate of Kerr blackhole:

We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass and the angular momentum of the blackhole. We consider the particle as an ellipsoid shell of energy ω and angular momentum ωa. When the particle's self-gravitation is taken into account, then equation (5.7.11) and (5.7.14) should be modified. To ensure the conservation of energy and angular momentum, we fix the total mass and angular momentum of the blackhole and allow the hole mass and angular momentum to fluctuate. When particle tunnels out , the blackhole mass and angular momentum will become $M-\omega$ and $a(M-\omega)$ respectively.

The shell of energy will move along the modified null geodesic in the radial direction

$$\dot{r}=\frac{\pm\rho^2-\sqrt{\rho^2(\rho^2-\bar{\Delta})}}{\sqrt{(r^2+a^2)^2-a^2\bar{\Delta}\sin^2\theta}} \quad \text{.............................(5.7.15)}$$

where $\bar{\Delta}=r^2-2(M-\omega)+a^2$ is the horizon equation after the emission of the particle with energy ω .

Now the coordinate ϕ does not appear in the dragged Painleve-Gullstrand Kerr metric (5.7.11) . So ϕ is an ignorable coordinate in the Lagrangian function L.. To eliminate this degree of freedom completely, the action should be written as

$$L=\int_{t_{in}}^{t_{out}}(L-P_\phi\dot{\phi})dt \quad \text{....................................(5.7.16)}$$

So the imaginary part of the action is

$$\operatorname{Im}S=\operatorname{Im}\int_{r_{in}}^{r_{out}}\left[\int_{(0,0)}^{(P_r,P_\phi)}(\dot{r}\,dP_r^{\prime}-\dot{\phi}\,dP_\phi^{\prime})\right]\frac{dr}{\dot{r}} \quad \text{...............................(5.7.17)}$$

where P_r and P_ϕ are two canonical momentum conjugate to r and ϕ respectively.

$r_{in}=M+\sqrt{M^2-a^2}$ and $r_{out}=(M-\omega)+\sqrt{(M-\omega)^2-a^2}$ are the locations of the event horizon before and after a particle tunnels out, they are just inside and outside the barrier through which the particle tunnels.

We now eliminate the momentum in favor of energy by using Hamilton's equations

$$\dot{r}=\frac{dH}{dP_r}\bigg|_{(r;\phi,P_\phi)}=\frac{d(M-\omega)}{dP_r} \quad \text{...........................(5.7.18)}$$

$$\dot{\phi}=\frac{dH}{dP_\phi}\bigg|_{(\phi;r,P_r)}=a\bar{\Omega}\frac{d(M-\omega)}{dP_\phi} \quad \text{....................(5.7.19)}$$

where $dH_{(\phi;r,P_r)}=\bar{\Omega}\,dJ=a\bar{\Omega}\,d(M-\omega)$ represents the energy change of the blackhole because of the loss of the angular momentum when a particle tunnels out [180] , and the dragging angular velocity is given by

$$\bar{\Omega} = \frac{a[(r^2+a^2)-\bar{\Delta}]}{(r^2+a^2)^2-\bar{\Delta}a^2\sin^2\theta} \qquad \text{..........................(5.7.20)}$$

Substituting equations (5.7.15), (5.7.18)and (5.7.19) into equation (5.7.17) and noting that we must choose the positive sign in equation (5.7.15) as the particle is propagating from inside to outside the horizon, then we have

$$\operatorname{Im} S = \operatorname{Im}[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}(1-a\Omega')\frac{dr}{\dot{r}}d(M-\omega')] = \operatorname{Im}[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{(1-a\Omega')\sqrt{(r^2+a^2)^2-a^2\Delta'\sin^2\theta}}{\rho^2-\sqrt{\rho^2(\rho^2-\Delta')}}drd(M-\omega'$$

..........................(5.7.21)

where

$$\Delta' = r^2+a^2-2(M-\omega')r = (r-r_+')(r-r_-')$$

$$r_\pm' = M-\omega' \pm \sqrt{(M-\omega')^2-a^2}$$

We multiply and divide the integrand with $\rho^2+\sqrt{\rho^2(\rho^2-\Delta')}$ to obtain

$$\operatorname{Im} S = \operatorname{Im}[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{(1-a\Omega')[\rho^2+\sqrt{\rho^2(\rho^2-\Delta')}]\sqrt{(r^2+a^2)^2-a^2\Delta'\sin^2\theta}}{\rho^2(r-r_+')(r-r_-')}drd(M-\omega')]$$

.....................................(5.7.22)

We see that $r = r_+'$ is a pole of order one. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\operatorname{Im} S = -2\pi\int_{M}^{M-\omega}\frac{(M-\omega')^2+(M-\omega')\sqrt{(M-\omega')^2-a^2}-\frac{1}{2}a^2}{\sqrt{(M-\omega')^2-a^2}}d(M-\omega')$$

..........................(5.7.23)

Finishing the integral we have

$$\operatorname{Im} S = \pi[M^2-(M-\omega)^2+M\sqrt{M^2-a^2}-(M-\omega)\sqrt{(M-\omega)^2-a^2}]$$

....................................(5.7.24)

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{-2\pi[M^2-(M-\omega)^2+M\sqrt{M^2-a^2}-(M-\omega)\sqrt{(M-\omega)^2-a^2}]}$$

..........................(5.7.25)

Using Bekenstein-Hawking entropy formula $S_{BH} = \pi(r_+^2+a^2)$,

we have $S_{BH}(M) = \pi[2M^2+2M\sqrt{M^2-a^2}-a^2]$

$$S_{BH}(M-\omega) = \pi[2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2}-a^2]$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$$
$$= 2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2} - M^2 - M\sqrt{M^2-a^2}]$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. From equation (5.7.25) we have

$\Gamma \sim e^{\Delta S_{BH}}$ ....(5.7.26)

From equation (5.7.26) we see that when the quadratic term neglected, then it reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature i.e. $e^{-\beta\omega}$ where $\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = \frac{2\pi\left[M+\sqrt{M^2-a^2}\right]^2}{\sqrt{M^2-a^2}}$. Also equation (5.7.26) indicates that when the energy conservation and the angular momentum conservation as well as the particle's self-gravitation are taken into account , the tunneling rate is related to the change of blackhole entropy during the process of the particle's emission and the radiant spectrum is not precisely thermal.

5.8 Uncharged particle tunneling from Kerr-NUT blackhole:

5.8.1 Kerr-NUT blackhole metric in dragging coordinate system and infinite red- shift surface of Kerr-NUT blackhole: The line element of Kerr-NUT blackhole is given by[177]

$$ds^2 = -\frac{\Delta^2}{\rho^2}(dt - P\,d\phi)^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(F+l^2)d\phi - a\,dt]^2$$
....(5.8.1)

where

$$\Delta^2 = r^2 - 2Mr + a^2 - l^2$$
$$\rho^2 = r^2 + (l + a\cos\theta)^2$$
$$P = a\sin^2\theta - 2l\cos\theta$$
$$F = r^2 + a^2$$
$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum and r is the radial distance from the center of the body , l is the NUT parameter . The equation of the event horizon is given by,

$\Delta^2 = 0$ which gives,

$r_\pm = M \pm \sqrt{M^2 - a^2 + l^2}$, $M^2 > a^2 - l^2$..............................(5.8.2)

To determine the area of the Kerr-NUT blackhole, we consider t is constant. At the event horizon $r = r_+$, the line element (5.8.1) can be written as

$$ds^2 = \rho^2 d\theta^2 - \frac{\Delta^2 P^2}{\rho^2}d\phi^2 + \frac{\sin^2\theta(r_+^2 + a^2 + l^2)^2}{\rho^2}d\phi^2 \text{ ....(5.8.3)}$$

The determinant of the two dimensional metric above is

$$g = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} = \sin^2\theta(r_+^{\ 2} + a^2 + l^2) \quad \text{.............................(5.8.4)}$$

So the area of the blackhole is

$$A = \int dA = \int \sqrt{g}\, d\theta d\phi = 4\pi(r_+^{\ 2} + a^2 + l^2) \quad \text{.........................(5.8.5)}$$

The Bekenstein-Hawking entropy is therefore

$$S_{BH} = \frac{A}{4} = \pi(r_+^{\ 2} + a^2 + l^2) \text{.......................................(5.8.6)}$$

The infinite red shift surface is given by $g_{00} = 0$ which gives

$$r_\pm = M \pm \sqrt{M^2 + l^2 - a^2\cos^2\theta} \quad \text{...............................(5.8.7)}$$

Obviously the infinite red shift surface does not coincide with the event horizon surface , which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Also there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (5.8.1) into a dragging coordinate system.

Let $\Omega = \dfrac{d\phi}{dt} = -\dfrac{g_{03}}{g_{00}}$ ....(5.8.8)

where Ω is the angular velocity.

For the metric (5.8.1) we have,

$$g_{00} = \frac{-(\Delta^2 - a^2\sin^2\theta)}{\rho^2},\; g_{11} = \frac{\rho^2}{\Delta^2},\; g_{22} = \rho^2,\; g_{33} = \frac{\sin^2\theta(F + l^2)^2 - \Delta^2 P^2}{\rho^2}$$

$$g_{03} = \frac{\Delta^2 P - a\sin^2\theta(F + l^2)}{\rho^2}$$

From (5.8.8), $\Omega = \dfrac{d\phi}{dt} = -\dfrac{g_{03}}{g_{00}} = -\dfrac{\Delta^2 P - a\sin^2\theta(F + l^2)}{\sin^2\theta(F + l^2)^2 - \Delta^2 P^2}$

.....................................(5.8.9)

At the horizon the angular velocity becomes,

$$\Omega_h = \frac{a}{r_+^2 + a^2 + l^2} \quad \text{...........................(5.8.10)}$$

The line element (5.8.1) in the dragging coordinate system becomes,

$$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dr^2 + g_{22} d\theta^2 \quad \text{..................(5.8.11)}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta^2 \rho^2 \sin^2\theta}{\sin^2\theta(r^2+a^2+l^2)^2 - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}$.

The line element (5.8.11) represents a 3-dimensional hypersurface of 4-dimensional spacetime. The infinite red-shift surface now coincide with the event horizon surface in the dragging coordinate system.

5.8.2 Painleve-like coordinate transformation and null geodesics of Kerr-NUT blackhole:

To investigate the Hawking radiation as tunneling process it is necessary to eliminate coordinate singularity at the event horizon. In the expression (5.8.11) , there still exists coordinate singularity at the event horizon in the dragging coordinate system. So we continue performing a general Painleve coordinate transformation[173]. For this transformation we set,

$$dt_{K-NUT} = dt + F(r,\theta)dr + G(r,\theta)d\theta \quad \text{.......................(5.8.12)}$$

where $F(r,\theta)$ and $G(r,\theta)$ are two determined functions of r and θ , and satisfy the integrability condition ,

$$\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r} \quad \text{.......................(5.8.13)}$$

Thus from (5.8.11) we obtain,

$$ds^2 = \hat{g}_{00}\,dt^2 + \left\{\hat{g}_{00}\,F^2(r,\theta) + g_{11}\right\}dr^2 + \left\{\hat{g}_{00}\,G^2(r,\theta) + g_{22}\right\}d\theta^2 + 2G(r,\theta)\,\hat{g}_{00}\,dtd\theta$$

$$+ 2F(r,\theta)G(r,\theta)\,\hat{g}_{00}\,drd\theta + 2F(r,\theta)\,\hat{g}_{00}\,dtdr \quad \text{.......................(5.8.14)}$$

We demand that constant time- slices are flat Euclidean space in radial. So we set,

$$\hat{g}_{00}\,F^2(r,\theta) + g_{11} = 1$$

$$\Rightarrow F(r,\theta) = \pm\sqrt{\frac{1-g_{11}}{\hat{g}_{00}}} \quad \text{.......................(5.8.15)}$$

From equation (5.8.13) , $G(r,\theta) = \int \frac{\delta F(r,\theta)}{\delta\theta}dr + C(\theta)$ ....(5.8.16)

where $C(\theta)$ is an arbitrary analytic function of θ .

Substituting the value of $F(r,\theta)$ into equation (5.8.14) we get,

$$ds^2 = \hat{g}_{00}\, dt^2 + dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2\sqrt{\hat{g}_{00}(1-g_{11})}\; G(r,\theta) dr d\theta$$

$$+ 2\hat{g}_{00}\, G(r,\theta)\, dt d\theta \pm 2\sqrt{\hat{g}_{00}(1-g_{11})}\;\; dt dr$$

$$\ldots\ldots\ldots\ldots\ldots\ldots..(5.8.17)$$

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon.

According to Landau's theory of the coordinate clock synchronization[174] in a spacetime decomposed in 3+1 dimension, the difference of coordinate times of two events taking place simultaneously in different place is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}} dx^i \qquad (i = 1,2,3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5.8.18)$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy

$$\frac{\delta}{\delta x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\delta}{\delta x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad , \; (i, j = 1,2,3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5.8.19)$$

Now the metric (5.8.17) in the new coordinate system , which we called the dragged Painleve-Gullstrand - Kerr-NUT metric, has a number of attractive features : (1) the metric is well-behaved at the event horizon; (2) the time coordinate t represents the local proper time for radially free-falling observers; (3) the hypersurfaces of constant time-slices are just flat Euclidean space in the oblate spheroidal coordinates; (4) by substituting the components of the metric (5.8.17) into equation (5.8.19), we see that the metric satisfy the Landau's condition of the coordinate clock synchronization $\frac{\delta F(r,\theta)}{\delta \theta} = \frac{\delta G(r,\theta)}{\delta r}$; (5) the infinite red-shift surface coincide with the event horizon surface so that the WKB approximation can be used. These attractive features are very advantageous for us to discuss Hawking radiation as tunneling and to do an explicit computation of the tunneling probability at the event horizon.

Now in order to investigate the tunneling process we evaluated the radial null geodesics described by equation (5.8.17) . Since the tunneling processes take place near the event horizon, so we may consider a particle tunneling from the event horizon as an ellipsoid shell . To conserve the symmetry of the spacetime , we think the particle should be still an ellipsoid shell during the tunneling process i.e. the particle does not have motion in

θ-direction[179]. Under these condition we obtain the radial null geodesics from equation (5.8.17) $(ds^2 = 0 = d\theta^2)$ as

$$\dot{r} = \frac{\sin\theta[\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta^2)}]}{\sqrt{\sin^2\theta(r^2 + a^2 + l^2)^2 - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}} \qquad \text{.................................(5.8.20)}$$

where a dot denotes differentiation with respect to t and the positive sign $(+)$ represents an outgoing geodesics and the negative sign $(-)$ represents an ingoing geodesics.

5.8.3 Tunneling rate of the Kerr-NUT blackhole:

We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass and the angular momentum of the blackhole. We consider the particle as an ellipsoid shell of energy ω and angular momentum ωa. When the particle's self-gravitation is taken into account, then equation (5.8.17) and (5.8.20) should be modified. To ensure the conservation of energy and angular momentum, we fix the total mass and angular momentum of the blackhole and allow the hole mass and angular momentum to fluctuate. When particle tunnels out , the blackhole mass and angular momentum will become $M - \omega$ and $a(M - \omega)$ respectively.

The shell of energy will move along the modified null geodesic in the radial direction

$$\dot{r} = \frac{\sin\theta[\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \bar{\Delta}^2)}]}{\sqrt{\sin^2\theta(r^2 + a^2 + l^2)^2 - \bar{\Delta}^2(a\sin^2\theta - 2l\cos\theta)^2}} \qquad \text{...............................(5.8.21)}$$

where $\bar{\Delta}^2 = r^2 - 2(M - \omega) + a^2 - l^2$ is the horizon equation after the emission of the particle with energy ω .

Now the coordinate ϕ does not appear in the dragged Painleve-Gullstrand Kerr-NUT metric (5.8.17) . So ϕ is an ignorable coordinate in the Lagrangian function L.. To eliminate this degree of freedom completely, the action should be written as

$$L = \int_{t_{in}}^{t_{out}} (L - P_\phi \dot{\phi}) dt \quad \text{.....................................(5.8.22)}$$

So the imaginary part of the action is

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} [\int_{(0,0)}^{(P_r, P_\phi)} (\dot{r}\, dP_r' - \dot{\phi}\, dP_\phi')] \frac{dr}{\dot{r}} \quad \text{..............................(5.8.23)}$$

where P_r and P_ϕ are two canonical momentum conjugate to r and ϕ respectively.

$r_{in} = M + \sqrt{M^2 - a^2 + l^2}$ and $r_{out} = (M - \omega) + \sqrt{(M - \omega)^2 - a^2 + l^2}$ are the locations of the event horizon before and after a particle tunnels out, they are just inside and outside the barrier through which the particle tunnels.

We now eliminate the momentum in favor of energy by using Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r} \Big|_{(r;\phi, P_\phi)} = \frac{d(M - \omega)}{dP_r} \quad \text{............................(5.8.24)}$$

$$\dot{\phi} = \frac{dH}{dP_\phi} \Big|_{(\phi; r, P_r)} = a\bar{\Omega} \frac{d(M - \omega)}{dP_\phi} \quad \text{.....................(5.8.25)}$$

where $dH_{(\phi; r, P_r)} = \bar{\Omega}\, dJ = a\bar{\Omega}\, d(M - \omega)$ represents the energy change of the blackhole because of the loss of the angular momentum when a particle tunnels out [180] , and the dragging angular velocity is given by

$$\bar{\Omega} = -\frac{\bar{\Delta}^2 P - a\sin^2\theta (r^2 + a^2 + l^2)}{\sin^2\theta (r^2 + a^2 + l^2)^2 - \bar{\Delta}^2 P^2} \quad \text{...........................(5.8.26)}$$

Substituting equations (5.8.21), (5.8.24) (5.8.25) and (5.8.26) into equation (5.8.23) and noting that we must choose the positive sign in equation (5.8.21) as the particle is propagating from inside to outside the horizon, then we have

$$\operatorname{Im} S = \operatorname{Im}[\int_{r_{in}}^{r_{out}} \int_{M}^{M - \omega} (1 - a\Omega') \frac{dr}{\dot{r}} d(M - \omega')$$

$$= \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M - \omega} (1 - a\Omega') \frac{\sqrt{(r^2 + a^2 + l^2)^2 \sin^2\theta - (\Delta')^2 (a\sin^2\theta - 2l\cos\theta)^2}}{\sin\theta [\rho^2 - \sqrt{\rho^2 \{\rho^2 - (\Delta')^2\}}} drd(M - \omega')$$

$$\text{.......................................(5.8.27)}$$

where

$$(\Delta')^2 = r^2 + a^2 - l^2 - 2(M-\omega')r = (r - r_+^/)(r - r_-^/)$$

$$r_\pm^/ = M - \omega' \pm \sqrt{(M-\omega')^2 - a^2 + l^2}$$

We multiply and divide the integrand with $\rho^2 + \sqrt{\rho^2\{\rho^2 - (\Delta')^2\}}$ to obtain

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M-\omega} (1 - a\Omega') \frac{[\rho^2 + \sqrt{\rho^2\{\rho^2 - (\Delta')^2\}}]\sqrt{(r^2 + a^2 + l^2)^2 \sin^2\theta - (\Delta')^2 (a \sin^2\theta - 2l\cos\theta}}{(r - r_+^/)(r - r_-^/)\rho^2 \sin\theta} drd(M-\omega')$$

....(5.8.28)

We see that $r = r_+^/$ is a pole of order one. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\operatorname{Im} S = -2\pi \int_{M}^{M-\omega} \frac{2[(M-\omega')^2 - a^2 + l^2] + a^2 + 2(M-\omega')\sqrt{(M-\omega')^2 - a^2 + l^2}}{\sqrt{(M-\omega')^2 - a^2 + l^2}} d(M-\omega')$$

................................(5.8.29)

Finishing the integral[178] we have

$$\operatorname{Im} S = -\pi[(M-\omega)^2 - M^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 + l^2} - M\sqrt{M^2 - a^2 + l^2}]$$

...............................(5.8.30)

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{2\pi[(M-\omega)^2 - M^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 + l^2} - M\sqrt{M^2 - a^2 + l^2}]}$$

.............................(5.8.31)

Using Bekenstein-Hawking entropy formula $S_{BH} = \pi(r_+^2 + a^2 + l^2)$,

we have $S_{BH}(M) = \pi[2M^2 + 2M\sqrt{M^2 - a^2 + l^2} - a^2 + l^2]$

$$S_{BH}(M-\omega) = \pi[2(M-\omega)^2 + 2(M-\omega)\sqrt{(M-\omega)^2 - a^2 + l^2} - a^2 + l^2]$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$$

$$= 2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 + l^2} - M^2 - M\sqrt{M^2 - a^2 + l^2}]$$

Where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. From equation (5.8.31) we have

$\Gamma \sim e^{\Delta S_{BH}}$ ....(5.8.32)

From equation (5.8.32) we see that when the quadratic term neglected, then it reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature i.e. $e^{-\beta\omega}$ where $\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = \frac{2\pi\left[M + \sqrt{M^2 - a^2 + l^2}\right]^2}{\sqrt{M^2 - a^2 + l^2}}$.

Also equation (5.8.32) indicates that when the energy conservation and the angular momentum conservation as well as the particle's self-gravitation are taken into account , the tunneling rate is related to the change of blackhole entropy during the process of the particle's emission and the radiant spectrum is not precisely thermal.

5.9 Uncharged particle tunneling from Kerr-Newmann Blackhole:

5.9.1 Dragged Painleve-Gullstrand Kerr-Newmann metric and null geodesics of Kerr-Newmann Blackhole:

The line element of Kerr-Newmann metric in the Boyer-Lindquist coordinate system is given by,

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a\, dt]^2 \qquad \text{..........................(5.9.1)}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$
$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum , Q is the electric charge. The equation of the event horizon is given by,

$\Delta = 0$ which gives, $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$, $M^2 > a^2 + Q^2$.

In order to investigate Hawking radiation as tunneling from Kerr-Newman blackhole we first adopt dragging coordinate system to overcome two difficulties. First , the event horizon $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$ does not coincide with the infinite red-shift surface $r_{\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta - Q^2}$, which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Second , as there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (5.9.1) into a dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ ....(5.9.2)

where Ω is the angular velocity.

For the metric (5.9.1) we have,

$$g_{00} = \frac{-(\Delta - a^2 \sin^2\theta)}{\rho^2},\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \frac{\sin^2\theta[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta]}{\rho^2}$$

$$g_{03} = \frac{-a\sin^2\theta[(r^2+a^2)-\Delta]}{\rho^2}$$

From (5.9.2), $$\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}} = \frac{a[(r^2+a^2)-\Delta]}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$$ ....(5.9.3)

At the horizon the angular velocity becomes,

$\Omega_h = \frac{a}{r_+^2 + a^2}$ ....(5.9.4)

The line element (5.9.1) in the dragging coordinate system becomes,

$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dt^2 + g_{22} d\theta^2$ ....(5.9.5)

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$.

The line element (5.9.5) represents a 3-dimensional hypersurface of 4-dimensional spacetime. Although in the dragging coordinate system does not has any singularity at the event horizon, and we can also get the Hawking pure thermal spectrum , yet this coordinate system is not what we want because it is not flat Euclidean space in radial to constant time slices. So we continue performing a general Painleve –Gullstrand coordinate transformation[173]. For this transformation we set,

$dt_{KN} = dt + F(r,\theta)dr + G(r,\theta)d\theta$ ....(5.9.6)

where $F(r,\theta)$ and $G(r,\theta)$ are two determined functions of r and θ , and satisfy the integrability condition ,

$\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$ ....(5.9.7)

Thus from (5.9.5) we obtain,

$$ds^2 = \hat{g}_{00}\, dt^2 + \left\{ \hat{g}_{00} F^2(r,\theta) + g_{11} \right\} dr^2 + \left\{ \hat{g}_{00} G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2G(r,\theta)\hat{g}_{00}\, dt d\theta$$

$$+ 2F(r,\theta)G(r,\theta)\hat{g}_{00}\, dr d\theta + 2F(r,\theta)\hat{g}_{00}\, dt dr \qquad \text{.................................(5.9.8)}$$

We demand that constant time- slices are flat Euclidean space in radial. So we set,

$$\hat{g}_{00} F^2(r,\theta) + g_{11} = 1 \qquad \text{...............................(5.9.9)}$$

$$\Rightarrow F(r,\theta) = \pm\sqrt{\frac{1-g_{11}}{\hat{g}_{00}}}$$

From equation (5.9.7) , $G(r,\theta) = \int \frac{\delta F(r,\theta)}{\delta\theta} dr + C(\theta)$ ....(5.9.10)

where $C(\theta)$ is an arbitrary analytic function of θ .
Substituting the value of $F(r,\theta)$ into equation (5.9.8) we get,

$$ds^2 = \hat{g}_{00}\, dt^2 + dr^2 + \left\{ \hat{g}_{00} G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2\sqrt{\hat{g}_{00}(1-g_{11})}\; G(r,\theta) dr d\theta$$

$$+ 2\hat{g}_{00}\, G(r,\theta)\, dt d\theta \pm 2\sqrt{\hat{g}_{00}(1-g_{11})}\;\, dt dr \qquad \text{...................(5.9.11)}$$

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon.
According to Landau's theory of the coordinate clock synchronization[174] in a spacetime decomposed in 3+1 dimension, the difference of coordinate times of two events taking place simultaneously in different place is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}} dx^i \qquad (i = 1,2,3) \qquad \text{.......................(5.9.12)}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy

$$\frac{\delta}{\delta x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\delta}{\delta x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad , \ (i,j = 1,2,3) \qquad \text{.......................(5.9.13)}$$

Now the metric (5.9.11) in the new coordinate system , which we called the dragged Painleve-Gullstrand Kerr-Newman metric, has a number of attractive features : (1) the metric is well-behaved at the event horizon; (2) the time coordinate t represents the local proper time for radially free-falling observers; (3) the hypersurfaces of constant time-slices are just flat

Euclidean space in the oblate spheroidal coordinates; (4) by substituting the components of the metric (5.9.11) into equation (5.9.13), we see that the metric satisfy the Landau's condition of the coordinate clock synchronization $\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$; (5) the infinite red-shift surface coincide with the event horizon surface so that the WKB approximation can be used. These attractive features are very advantageous for us to discuss Hawking radiation as tunneling and to do an explicit computation of the tunneling probability at the event horizon.

Now in order to investigate the tunneling process we evaluated the radial null geodesics described by equation (5.9.11) . Since the tunneling processes take place near the event horizon, so we may consider a particle tunneling from the event horizon as an ellipsoid shell . To conserve the symmetry of the spacetime , we think the particle should be still an ellipsoid shell during the tunneling process i.e. the particle does not have motion in θ-direction[179]. Under these condition we obtain the radial null geodesics from equation (5.9.11) $(ds^2 = 0 = d\theta^2)$ as

$$\dot{r} = \frac{\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta)}}{\sqrt{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta}} \qquad \text{.......................(5.9.14)}$$

where a dot denotes differentiation with respect to t and the positive sign $(+)$ represents an outgoing geodesics and the negative sign $(-)$ represents an ingoing geodesics.

5.9.2 Tunneling rate of Kerr-Newman blackhole:

We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass and the angular momentum of the blackhole. We consider the particle as an ellipsoid shell of energy ω and angular momentum ωa. When the particle's self-gravitation is taken into account, then equation (5.9.11) and (5.9.14) should be modified. To ensure the conservation of energy and angular momentum, we fix the total mass and angular momentum of the blackhole and allow the hole mass and angular momentum to fluctuate. When particle tunnels out , the blackhole mass and angular momentum will become $M - \omega$ and $a(M - \omega)$ respectively.

The shell of energy will move along the modified null geodesic in the radial direction

$$\dot{r} = \frac{\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \bar{\Delta})}}{\sqrt{(r^2+a^2)^2 - a^2\bar{\Delta}\sin^2\theta}} \quad \text{.........................(5.9.15)}$$

where $\bar{\Delta} = r^2 - 2(M-\omega) + a^2 + Q^2$ is the horizon equation after the emission of the particle with energy ω .

Now the coordinate ϕ does not appear in the dragged Painleve-Gullstrand Kerr-Newman metric (5.9.11) . So ϕ is an ignorable coordinate in the Lagrangian function L.. To eliminate this degree of freedom completely, the action should be written as

$$L = \int_{t_{in}}^{t_{out}} (L - P_\phi \dot{\phi})dt \quad \text{..................................(5.9.16)}$$

So the imaginary part of the action is

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} [\int_{(0,0)}^{(P_r,P_\phi)} (\dot{r}\, dP_r' - \dot{\phi}\, dP_\phi')]\frac{dr}{\dot{r}} \quad \text{............................(5.9.17)}$$

where P_r and P_ϕ are two canonical momentum conjugate to r and ϕ respectively.

$r_{in} = M + \sqrt{M^2 - a^2 - Q^2}$ and $r_{out} = (M-\omega) + \sqrt{(M-\omega)^2 - a^2 - Q^2}$ are the locations of the event horizon before and after a particle tunnels out, they are just inside and outside the barrier through which the particle tunnels.

We now eliminate the momentum in favor of energy by using Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;\phi,P_\phi)} = \frac{d(M-\omega)}{dP_r} \quad \text{..........................(5.9.18)}$$

$$\dot{\phi} = \frac{dH}{dP_\phi}\Big|_{(\phi;r,P_r)} = a\bar{\Omega}\frac{d(M-\omega)}{dP_\phi} \quad \text{....................(5.9.19)}$$

where $dH_{(\phi;r,P_r)} = \bar{\Omega}\, dJ = a\bar{\Omega}\, d(M-\omega)$ represents the energy change of the blackhole because of the loss of the angular momentum when a particle tunnels out [180] , and the dragging angular velocity is given by

$$\bar{\Omega} = \frac{a[(r^2+a^2) - \bar{\Delta}]}{(r^2+a^2)^2 - \bar{\Delta}a^2\sin^2\theta} \quad \text{..........................(5.9.20)}$$

Substituting equations (5.9.15), (5.9.18)and (5.9.19) into equation (5.9.17) and noting that we must choose the positive sign in equation (5.9.15) as the particle is propagating from inside to outside the horizon, then we have

$$\operatorname{Im} S = \operatorname{Im}\left[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}(1-a\Omega')\frac{dr}{\dot{r}}d(M-\omega')\right] = \operatorname{Im}\left[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{(1-a\Omega')\sqrt{(r^2+a^2)^2-a^2\Delta'\sin^2\theta}}{\rho^2-\sqrt{\rho^2(\rho^2-\Delta')}}drd(M-\omega\right.$$

............................(5.9.21)

where

$$\Delta' = r^2+a^2+Q^2-2(M-\omega')r = (r-r_+')(r-r_-')$$

$$r_\pm' = M-\omega' \pm \sqrt{(M-\omega')^2-a^2-Q^2}$$

We multiply and divide the integrand with $\rho^2+\sqrt{\rho^2(\rho^2-\Delta')}$ to obtain

$$\operatorname{Im} S = \operatorname{Im}\left[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{(1-a\Omega')[\rho^2+\sqrt{\rho^2(\rho^2-\Delta')}]\sqrt{(r^2+a^2)^2-a^2\Delta'\sin^2\theta}}{\rho^2(r-r_+')(r-r_-')}drd(M-\omega')\right]$$

......................................(5.9.22)

We see that $r=r_+'$ is a pole of order one. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\operatorname{Im} S = -2\pi\int_{M}^{M-\omega}\frac{(M-\omega')^2+(M-\omega')\sqrt{(M-\omega')^2-a^2-Q^2}-\frac{1}{2}(a^2+Q^2)}{\sqrt{(M-\omega')^2-a^2-Q^2}}d(M-\omega')$$

..............................(5.9.23)

Finishing the integral we have

$$\operatorname{Im} S = \pi[M^2-(M-\omega)^2+M\sqrt{M^2-a^2-Q^2}-(M-\omega)\sqrt{(M-\omega)^2-a^2-Q^2}\,]$$

..............................(5.9.24)

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{-2\pi[M^2-(M-\omega)^2+M\sqrt{M^2-a^2-Q^2}-(M-\omega)\sqrt{(M-\omega)^2-a^2-Q^2}]}$$

..............................(5.9.25)

Using Bekenstein-Hawking entropy formula $S_{BH}=\pi(r_+^2+a^2)$,

we have

$$S_{BH}(M)=\pi[2M^2+2M\sqrt{M^2-a^2-Q^2}-a^2-Q^2]$$

$$S_{BH}(M-\omega)=\pi[2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2-Q^2}-a^2-Q^2]$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$$
$$= 2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 - Q^2} - M^2 - M\sqrt{M^2 - a^2 - Q^2}]$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. From equation (5.9.25) we have

$\Gamma \sim e^{\Delta S_{BH}}$ ....(5.9.26)

From equation (5.9.26) we see that when the quadratic term neglected, then it reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature i.e. $e^{-\beta\omega}$ where $\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = \frac{2\pi\left[M + \sqrt{M^2 - a^2 - Q^2}\right]^2}{\sqrt{M^2 - a^2 - Q^2}}$.

Also equation (5.9.26) indicates that when the energy conservation and the angular momentum conservation as well as the particle's self-gravitation are taken into account , the tunneling rate is related to the change of blackhole entropy during the process of the particle's emission and the radiant spectrum is not precisely thermal.

5.10 Uncharged particle tunneling from BTZ blackhole:

5.10.1 BTZ blackhole metric:

The BTZ blackhole metric is given by,

$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2[N^\phi(r)dt + d\phi]^2$ ....(5.10.1)

with cosmological constant $\Lambda = -l^{-2}$.

Here the squared lapse $N^2(r)$ and the angular shift $N^\phi(r)$ are given by

$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$, $N^\phi(r) = -\frac{J}{2r^2}$ ....(5.10.2)

where $-\infty < t < \infty$, $0 < r < \infty$ and $0 \le \varphi \le 2\pi$ and M, J denotes the ADM mass and angular momentum of the BTZ blackhole respectively (The BTZ unit $G_3 = \frac{1}{8}$ is adopted throughout this section).

The equation of the event horizons given by

$N^2(r) = 0$ ....(5.10.3)

which gives

$r_\pm^2 = l^2 \dfrac{[M \pm \sqrt{M^2 - \frac{J^2}{l^2}}]}{2}$ ....(5.10.4) with $M > 0$ *and* $|J| \le Ml$.

The area of the outer event horizon $A_h = 2\pi r_+$ and Bekenstein-Hawking entropy of the spinning BTZ blackhole is twice the perimeter L of the event horizon[40]

$S_{BH}(M,J,l) = 2L = 4\pi r_+$ ....(5.10.5)

5.10.2 Dragged Painleve-Gullstrand BTZ metric and null geodesics of BTZ blackhole:

In order to investigate Hawking radiation as tunneling from BTZ blackhole we first adopt dragging coordinate system to overcome two difficulties. First, the event horizon $r_\pm = \left\{ l^2 \dfrac{[M \pm \sqrt{M^2 - \frac{J^2}{l^2}}]}{2} \right\}^{\frac{1}{2}}$ does not coincide with the infinite red-shift surface $r_\pm = \pm l\sqrt{M}$, which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Second, as there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (5.10.1) into a dragging coordinate system.

Let $\Omega = \dfrac{d\phi}{dt} = -\dfrac{g_{03}}{g_{00}} = \dfrac{J}{2r^2}$ ....(5.10.6)

where Ω is the angular velocity. The line element (5.10.1) becomes

$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2$ ....(5.10.7)

In fact the line element (5.10.7) represents a 2 -dimensional hypersurface of 3-dimensional BTZ spacetime. This dragging coordinate system is not what we want to use for resolving tunneling effect in BTZ spacetime. So we need another coordinate transformation to make none of the components of either the metric or contra metric diverge at the horizon. Moreover constant time slices are just flat Euclidean in radial. To obtain a coordinate system analogous to Painleve-Gullstrand coordinate[173], we should perform a coordinate transformation

$dt_{BTZ} = dt + f(r)dr$ ....(5.10.8)

where $f(r)$ is a function of r, independent of t.

Putting equation (5.10.8) into(5.10.7) we obtain

$ds^2 = -N^2(r)dt^2 - 2N^2(r)f(r)dtdr + [N^{-2}(r) - N^2(r)f^2(r)]dr^2$ ....(5.10.9)

As a corollary, we demand that the metric (5.10.9) is flat Euclidean in radial to the constant time slices. We then get the condition

$$N^{-2}(r) - N^2(r) f^2(r) = 1$$

$$f(r) = \pm \frac{\sqrt{1 - N^2(r)}}{N^2(r)} \qquad \text{.............................(5.10.10)}$$

So the equation (5.10.9) becomes

$$ds^2 = -N^2(r)dt^2 \pm 2\sqrt{1 - N^2(r)}\ dtdr + dr^2 \qquad \text{.................(5.10.11)}$$

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon. There is now no singularity at the event horizon and the true character of the spacetime as being stationary but not static , is manifest. Also the infinite red shift surface coincide with the event horizon surface in the new line element which shall be referred to as a dragged Painleve –BTZ metric.

The radial null geodesics given by (putting $ds^2 = 0$)

$$\dot{r} = \pm 1 - \sqrt{1 - N^2(r)} \qquad \text{...............................(5.10.12)}$$

where a dot denotes differentiation with respect to t and the positive sign $(+)$ represents an outgoing geodesics and the negative sign $(-)$ represents an ingoing geodesics respectively under the assumption that t increase towards future.

5.10.3 Tunneling rate of BTZ blackhole:

Let us now focus on a semi-classical treatment of the associated radiation (outgoing massless particle). We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle tunnels out from the event horizon and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass .In the case of the rotating BTZ blackhole with fixed angular momentum J , the emitted particle is simply visualized as a shell of energy ω. When the particle's self-gravitation is taken into account, then equation (5.10.11) and (5.10.12) should be modified. To ensure the conservation of energy , we must fix the total ADM mass and let the ADM mass M of the BTZ blackhole vary. If a shell of energy ω is radiated outwards the outer horizon , then the BTZ blackhole mass will be reduced to $M - \omega$, so the line element will be modified to

$$ds^2 = -\bar{N}^2(r)dt^2 \pm 2\sqrt{1-\bar{N}^2(r)}\ dtdr + dr^2 \text{ ....(5.10.13)}$$

The shell of energy will move along the modified null geodesic in the radial direction

$$\dot{r} = \pm 1 - \sqrt{1-\bar{N}^2(r)} \text{ ....(5.10.14)}$$

where $\bar{N}^2(r) = -(M-\omega) + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$ is the horizon equation after the emission of the particle with energy ω.

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} is expressed as,

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} P_r\, dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{P_r} dP_r'\, dr \text{ ....(5.10.15)}$$

where r_{in} and r_{out} are the respective initial and final radii of the blackhole, P_r is the canonical momentum conjugate to r. The trajectory between these two radii is the barrier that the particle must tunnel through and

$$r_{in}^2 = l^2 \frac{[M + \sqrt{M^2 - \frac{J^2}{l^2}}]}{2} \text{ ....(5.10.16)}$$

$$r_{out}^2 = l^2 \frac{[(M-\omega) + \sqrt{(M-\omega)^2 - \frac{J^2}{l^2}}}{2}$$

By applying Hamilton's equation we change the momentum variable to the energy variable as

$$\dot{r} = \frac{dH}{dP_r} = \frac{d(M-\omega)}{dP_r} \text{ ....(5.10.17)}$$

From (5.10.15) we obtain

$$\operatorname{Im} S = \operatorname{Im} \int_{M_{in}}^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{d(M-\omega')dr}{1 - \sqrt{1 + (M-\omega') - \frac{r^2}{l^2} - \frac{J^2}{4r^2}}} \text{ ....(5.10.18)}$$

The integral can be done by deforming the contour, so as to ensure that the positive energy solutions decay in time (i.e. into the lower half of ω' plane). Doing the ω' integral first we obtain,

$$\operatorname{Im} S = -2\pi \int_{r_{in}}^{r_{out}} dr \text{ ....(5.10.19)}$$

Finishing the integral we have

$$\mathrm{Im}\,S = 2\pi[r_{in} - r_{out}] \quad \text{.........................(5.10.20)}$$

The tunneling rate is therefore

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} = e^{4\pi(r_{out}-r_{in})} = e^{\Delta S_{BH}} \quad \text{.........................(5.10.21)}$$

where $\Delta S_{BH} = S_{BH}(M-\omega,J,l) - S_{BH}(M,J,l)$ is the difference of the Bekenstein-Hawking entropies of the BTZ blackhole before and after the emission of the shell of energy ω.

It should be noted that the above discussion is limited to the case of energy conservation only. If we consider a spinning BTZ blackhole , so the rotation degree of freedom should be taken into account. This can be done by considering the emitted massless particle as a shell of energy ω and angular momentum j. Now taken into account not only the energy conservation but also the angular momentum conservation, we must fix the total ADM mass M and total angular momentum J, but let the ADM mass M and angular momentum J of the BTZ blackhole vary. If a shell of energy ω and angular momentum j tunnels out from the event horizon, then the BTZ blackhole mass and angular momentum will be reduced to $M-\omega$ and $J-j$ respectively. Therefore the modified line element and modified null geodesics are given by

$$ds^2 = -\bar{N}^2(r)dt^2 \pm 2\sqrt{1-\bar{N}^2(r)}\;dtdr + dr^2 \quad \text{.....................(5.10.22)}$$

and

$$\dot{r} = \pm 1 - \sqrt{1-\bar{N}^2(r)} \quad \text{..............................(5.10.23)}$$

where $\bar{N}^2(r) = -(M-\omega) + \frac{r^2}{l^2} + \frac{(J-j)^2}{4r^2}$ is the horizon equation after the emission of the particle with energy ω and angular momentum j .

Now the imaginary part of the action should be written as

$$\mathrm{Im}\,S = \mathrm{Im}[\int_{r_{in}}^{r_{out}} P_r dr - \int_{\phi_{in}}^{\phi_{out}} P_\phi d\phi]$$

$$= \mathrm{Im}\int_{r_{in}}^{r_{out}} [\int_0^{P_r} dP_r^{/} dr - \int_0^{P_\phi} dP_\phi^{/} d\phi]$$

$$= \mathrm{Im}\int_{r_{in}}^{r_{out}} [\int_{(0,0)}^{(P_r,P_\phi)} (\dot{r}\, dP_r^{/} - \dot{\phi}\, dP_\phi^{/})] \frac{dr}{\dot{r}}$$

$$\text{.............................(5.10.24)}$$

where P_r and P_ϕ are two canonical momentum conjugate to r and ϕ respectively.
To remove the momentum in favor of energy , we use the Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;\phi,P_\phi)} = \frac{d(M-\omega)}{dP_r}$$
$$\dot{\phi} = \frac{dH}{dP_\phi}\Big|_{(\phi;r,P_r)} = \bar{\Omega}\frac{d(J-j)}{dP_\phi} \qquad \text{.........................(5.10.25)}$$

where $dH_{(\phi;r,P_r)} = \bar{\Omega}\, d(J-j)$ represents the energy change of the blackhole because of the loss of the angular momentum when a particle tunnels out and the dragging angular velocity is given by

$$\bar{\Omega} = \frac{J-j}{2r^2} \qquad \text{.................................(5.10.26)}$$

Substituting equations (5.10.23),(5.10.25) and (5.10.26) into equation (5.10.24) we have

$$\operatorname{Im} S = \operatorname{Im}\int_{r_{in}}^{r_{out}(M-\omega,J-j)}\int_{(M,J)} [d(M-\omega') - \Omega' d(J-j')]\frac{dr}{\dot{r}}$$
$$= \operatorname{Im}\int_{r_{in}}^{r_{out}(M-\omega,J-j)}\int_{(M,J)} \frac{1+\sqrt{1-N'^2}}{N'^2}[d(M-\omega') - \frac{J-j'}{2r^2}d(J-j')]dr$$

.........................(5.10.27)

Finishing the integral we have

$$\operatorname{Im} S = 2\pi[r_{in} - r_{out}]$$

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{4\pi(r_{out}-r_{in})} = e^{\Delta S_{BH}} \qquad \text{...........................(5.10.28)}$$

Reproduces the Krause-Parikh-Wilczek's standard result for the tunneling picture, where $\Delta S_{BH} = S_{BH}(M-\omega, J-j, l) - S_{BH}(M, J, l)$ is the difference of the Bekenstein-Hawking entropies of the BTZ blackhole before and after the emission of the shell of energy ω and angular momentum j.

5.11 Uncharged particle tunneling from non-accelerating and rotating blackholes with electric and magnetic charges:

5.11.1 Non-accelerating and rotating blachkholes with electric and magnetic

charge: The Plebanski-Demianski [169,170,171] metric covers a large family of spacetimes which include, among others, the well known blackhole solutions like Schwartzschild, Reissner-Nordström , Kerr, Kerr-

Newman, Kerr-NUT, Kerr-Newman-NUT and many others. Here we study a special case of this family – blackholes with rotation but non-accelerating with electric and magnetic charges. The metric of this such kind of blackhole is given by [172]

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\rho^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]}{\rho^2}d\phi^2$$

$$-\frac{2a\sin^2\theta[(r^2+a^2)-\Delta]}{\rho^2}dtd\phi$$

....(5.11.1)

where $\Delta = r^2 + a^2 + e^2 + g^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$. Here M is the mass of the blackhole, e and g are the electric and magnetic charges respectively, a is the angular momentum per unit mass. The event horizon equations are given by $\Delta = 0$ which gives

$$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2 - g^2} \quad \text{......................(5.11.2)}$$

The event horizon area of this blackhole is given by

$$A = 4\pi(r_+^2 + a^2) \quad \text{.............................(5.11.3)}$$

and Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4} = \pi(r_+^2 + a^2) = \pi[2M^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2} - e^2 - g^2]$$

..........................(5.11.4)

5.11.2 Dragging coordinate system and infinite red- shift surface of non-accelerating and rotating blackholes with electric and magnetic charges:

The infinite red shift surface is given by $g_{00} = 0$ which gives

$$r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta - e^2 - g^2} \quad \text{................................(5.11.5)}$$

Obviously the infinite red shift surface does not coincide with the event horizon surface , which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Also there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (5.11.1) into a dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ ....(5.11.6)

where Ω is the angular velocity.

For the metric (5.11.1) we have,

$$g_{00} = \frac{-(\Delta - a^2 \sin^2\theta)}{\rho^2},\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]}{\rho^2}$$

$$g_{03} = \frac{-a\sin^2\theta[(r^2+a^2)-\Delta]}{\rho^2}$$

From (5.11.6), $\Omega = \frac{d\phi}{dt} = \frac{a[(r^2+a^2)-\Delta]}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}$ ....(5.11.7)

At the horizon the angular velocity becomes,

$$\Omega_h = \frac{a}{r_+^2 + a^2} \quad \text{.............................(5.11.8)}$$

The line element (5.11.1) in the dragging coordinate system becomes,

$$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dr^2 + g_{22} d\theta^2 \quad \text{...................(5.11.9)}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]}$.

The line element (5.11.9) represents a 3-dimensional hypersurface of 4-dimensional spacetime. The infinite red-shift surface now coincide with the event horizon surface in the dragging coordinate system. So the geometrical optical limit can be applied now.

5.11.3 Painleve-like coordinate transformation and null geodesics of non-accelerating and rotating blackholes with electric and magnetic charges:

To investigate the Hawking radiation as tunneling process it is necessary to eliminate coorditanate singularity at the event horizon. In the expression (5.11.9) , there still exists coordinate singularity at the event horizon in the dragging coordinate system. So we continue performing a general Painleve coordinate transformation[173]. This transformation can done by

$$dt \to dt + F(r,\theta)dr + G(r,\theta)d\theta \quad \text{...............................(5.11.10)}$$

where $F(r,\theta)$ and $G(r,\theta)$ are two determined functions of r and θ , and satisfy the integrability condition ,

$$\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r} \quad \text{........................(5.11.11)}$$

Thus from (5.11.9) we obtain,

$$ds^2 = \hat{g}_{00}\, dt^2 + \left\{ \hat{g}_{00}\, F^2(r,\theta) + g_{11} \right\} dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2G(r,\theta)\, \hat{g}_{00}\, dt d\theta$$

$$+ 2F(r,\theta) G(r,\theta)\, \hat{g}_{00}\, dr d\theta + 2F(r,\theta)\, \hat{g}_{00}\, dt dr$$

................................(5.11.12)

We demand that constant time- slices are flat Euclidean space in radial. So we set,

$$\hat{g}_{00}\, F^2(r,\theta) + g_{11} = 1$$

$$\Rightarrow F(r,\theta) = \pm \sqrt{\frac{1 - g_{11}}{\hat{g}_{00}}} \qquad \text{..............................(5.11.13)}$$

From equation (5.11.11), $G(r,\theta) = \int \frac{\delta F(r,\theta)}{\delta\theta} dr + C(\theta)$ ....(5.11.14)

where $C(\theta)$ is an arbitrary analytic function of θ .

Substituting the value of $F(r,\theta)$ into equation (5.11.12) we get,

$$ds^2 = \hat{g}_{00}\, dt^2 + dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2\sqrt{\hat{g}_{00}(1 - g_{11})}\; G(r,\theta) dr d\theta$$

$$+ 2\hat{g}_{00}\, G(r,\theta)\, dt d\theta \pm 2\sqrt{\hat{g}_{00}(1 - g_{11})}\;\, dt dr$$

....................(5.11.15)

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon.

According to Landau's theory of the coordinate clock synchronization[174] in a spacetime decomposed in 3+1 dimension, the difference of coordinate times of two events taking place simultaneously in different place is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}} dx^i \qquad (i = 1,2,3) \qquad \text{........................(5.11.16)}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy

$$\frac{\delta}{\delta x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\delta}{\delta x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad , \ (i,j = 1,2,3) \qquad \text{......................(5.11.17)}$$

Now the metric (5.11.15) in the new coordinate system , has a number of attractive features : (1) the metric is well-behaved at the event horizon; (2) the time coordinate t represents the local proper time for radially free-falling observers; (3) the hypersurfaces of constant time-slices are just flat Euclidean space in the oblate spheroidal coordinates; (4) by substituting the components of the metric (5.11.15) into equation (5.11.17), we see that the metric satisfy the Landau's condition of the coordinate clock synchronization $\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$; (5) the infinite red-shift surface coincide with the event horizon surface so that the WKB approximation can be used. These attractive features are very advantageous for us to discuss Hawking radiation as tunneling and to do an explicit computation of the tunneling probability at the event horizon.

Now in order to investigate the tunneling process we evaluated the radial null geodesics described by equation (5.11.15) . Since the tunneling processes take place near the event horizon, so we may consider a particle tunneling from the event horizon as an ellipsoid shell . To conserve the symmetry of the spacetime , we think the particle should be still an ellipsoid shell during the tunneling process i.e. the particle does not have motion in θ-direction[18]. Under these condition we obtain the radial null geodesics from equation (5.11.15) $(ds^2 = 0 = d\theta^2)$ as

$$\dot{r} = \frac{[\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta)}]}{\sqrt{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta}} \quad \text{.......................(5.11.18)}$$

where a dot denotes differentiation with respect to t and the positive sign $(+)$ represents an outgoing geodesics and the negative sign $(-)$ represents an ingoing geodesics.

5.11.4 Tunneling rate of non-accelerating and rotating blackholes with electric and magnetic charges:

We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass and the angular momentum of the blackhole. We consider the particle as an ellipsoid shell of energy ω and angular momentum ωa. When the particle's self-gravitation is taken into account, then equation (5.11.15) and (5.11.18) should be modified. To ensure the conservation of energy and angular momentum, we fix the total mass and angular momentum of the blackhole and allow the hole mass and angular momentum to fluctuate. When particle tunnels out , the blackhole mass and angular momentum will become $M - \omega$ and $a(M - \omega)$ respectively.

The shell of energy will move along the modified null geodesic in the radial direction

$$\dot{r} = \frac{[\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \bar{\Delta})}]}{\sqrt{(r^2+a^2)^2 - a^2\bar{\Delta}\sin^2\theta}} \quad \text{.........................(5.11.19)}$$

where $\bar{\Delta} = r^2 - 2(M-\omega) + a^2 + e^2 + g^2$ is the horizon equation after the emission of the particle with energy ω .

Now the coordinate ϕ does not appear in the dragged Painleve-Gullstrand metric (5.11.15) . So ϕ is an ignorable coordinate in the Lagrangian function L.. To eliminate this degree of freedom completely, the action should be written as

$$L = \int_{t_{in}}^{t_{out}} (L - P_\phi \dot{\phi})dt \quad \text{.................................(5.11.20)}$$

So the imaginary part of the action is

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} [\int_{(0,0)}^{(P_r,P_\phi)} (\dot{r}\,dP_r' - \dot{\phi}\,dP_\phi')] \frac{dr}{\dot{r}} \quad \text{.............................(5.11.21)}$$

where P_r and P_ϕ are two canonical momentum conjugate to r and ϕ respectively.

$r_{in} = M + \sqrt{M^2 - a^2 - e^2 - g^2}$ and $r_{out} = (M-\omega) + \sqrt{(M-\omega)^2 - a^2 - e^2 - g^2}$ are the locations of the event horizon before and after a particle tunnels out, they are just inside and outside the barrier through which the particle tunnels.

We now eliminate the momentum in favor of energy by using Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;\phi,P_\phi)} = \frac{d(M-\omega)}{dP_r} \quad \text{.........................(5.11.22)}$$

$$\dot{\phi} = \frac{dH}{dP_\phi}\Big|_{(\phi;r,P_r)} = a\bar{\Omega}\frac{d(M-\omega)}{dP_\phi} \quad \text{...................(5.11.23)}$$

where $dH_{(\phi;r,P_r)} = \bar{\Omega}\,dJ = a\bar{\Omega}\,d(M-\omega)$ represents the energy change of the blackhole because of the loss of the angular momentum when a particle tunnels out, and the dragging angular velocity is given by

$$\bar{\Omega} = \frac{a[(r^2+a^2) - \bar{\Delta}]}{(r^2+a^2)^2 - a^2\bar{\Delta}\sin^2\theta} \quad \text{.........................(5.11.24)}$$

Substituting equations (5.11.19), (5.11.22) (5.11.23) and (5.11.24) into equation (5.11.21) and noting that we must choose the positive sign in equation (5.11.19) as the particle is propagating from inside to outside the horizon, then we have

$$\operatorname{Im} S = \operatorname{Im}[\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}(1-a\Omega')\frac{dr}{\dot{r}}d(M-\omega')$$

$$= \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}(1-a\Omega')\ \frac{\sqrt{(r^2+a^2)^2-a^2\Delta'\sin^2\theta}}{[\rho^2-\sqrt{\rho^2(\rho^2-\Delta')}}\ drd(M-\omega')$$

$$\dots\dots\dots\dots(5.11.25)$$

where

$$\bar{\Delta} = r^2+a^2+e^2+g^2-2(M-\omega')r = (r-r_+^{/})(r-r_-^{/})$$

$$r_{\pm}^{/} = (M-\omega')\pm\sqrt{(M-\omega')^2-a^2-e^2-g^2}$$

We multiply and divide the integrand with $\rho^2+\sqrt{\rho^2(\rho^2-\Delta')}$ to obtain

$$\operatorname{Im} S = \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}(1-a\Omega')\ \frac{[\rho^2+\sqrt{\rho^2(\rho^2-\Delta')}]\sqrt{(r^2+a^2)^2-a^2\Delta'\sin^2\theta}}{(r-r_+^{/})(r-r_-^{/})\rho^2}\ drd(M-\omega')$$

$$\dots\dots\dots\dots(5.11.26)$$

We see that $r = r_+^{/}$ is a pole of order one. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\operatorname{Im} S = -2\pi\int_{M}^{M-\omega}\frac{(M-\omega')^2-\frac{1}{2}(a^2+e^2+g^2)+(M-\omega')\sqrt{(M-\omega')^2-a^2-e^2-g^2}}{\sqrt{(M-\omega')^2-a^2-e^2-g^2}}\ d(M-\omega')$$

$$\dots\dots\dots\dots(5.11.27)$$

Finishing the integral we have

$$\operatorname{Im} S = -\pi[(M-\omega)^2-M^2+(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2}-M\sqrt{M^2-a^2-e^2-g^2}]$$

$$\dots\dots\dots\dots(5.11.28)$$

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{2\pi[(M-\omega)^2-M^2+(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^{2^2}}-M\sqrt{M^2-a^2-e^2-g^{2^2}}]}$$

$$\text{.......... ....(5.11.29)}$$

Using Bekenstein-Hawking entropy formula $S_{BH} = \pi(r_+^2 + a^2)$,

we have $S_{BH}(M) = \pi[2M^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2} - e^2 - g^2]$

$$S_{BH}(M-\omega) = \pi[2(M-\omega)^2 + 2(M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2} - e^2 - g^2]$$

$$\begin{aligned}\Delta S_{BH} &= S_{BH}(M-\omega) - S_{BH}(M) \\ &= 2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2} - M^2 - M\sqrt{M^2 - a^2 - e^2 - g^2}]\end{aligned}$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. From equation (5.11.29) we have

$$\Gamma \sim e^{\Delta S_{BH}} \quad \text{..................................(5.11.30)}$$

The result is obviously consists with an underlying unitary theory. Following the reference [18], expanding ΔS_{BH} in $(\omega - \omega_0)$ and neglecting the higher order terms we have

$$\Gamma \sim e^{\Delta S_{BH}} = e^{-\beta(\omega-\omega_0)[1-\frac{r_+^2+a^2}{r_+^4}(M+\sqrt{M^2-a^2-e^2-g^2}-\frac{M(a^2+e^2+g^2)}{2(M^2-a^2-e^2-g^2)}(\omega-\omega_0)]} \quad \text{...............(5.11.31)}$$

where $\omega_0 = a\omega\Omega$ and β is the inverse Hawking temperature given by

$\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = \frac{2\pi\left[M + \sqrt{M^2 - a^2 - e^2 - g^2}\right]^2}{\sqrt{M^2 - a^2 - e^2 - g^2}}$. Also in equation (5.11.29) we see that the first term gives the thermal Boltzmann factors $e^{-\beta(\omega-\omega_0)}$ for the emanating radiation. The second term represents correction from the responds of the background geometry to the emission of a quantum. Furthermore equation (5.11.30) indicates that when the energy conservation and the angular momentum conservation as well as the particle's self-gravitation are taken into account , the tunneling rate is related to the change of blackhole entropy during the process of the particle's emission and the radiant spectrum is not precisely thermal.

5.11.5 Concluding remarks:

In this section, we have presented the Hawking radiation as tunneling from non-accelerating and rotating with electric and magnetic charged blackhole by applying Krause-Parikh-Wilczek's semi-classical quantum tunneling method[13,14,15,16]. We find that the emission rate at the event horizon is equal to the difference of Bekenstein-Hawking entropy before and after the emission of a particle. The Hawking temperature of this type of blackhole recovered by expanding ΔS_{BH} in $(\omega-\omega_0)$ and neglecting the higher order terms.

In special case , if we put $a=e=g=0$ then the result reduces for the Schwarzschild blackhole and if $a=o,\ g=0$ then the result reduces to the Reissner-Nordstrom blackhole and supports the Parikh-Wilczek's result[14]. Also if we assume the equivalent charge $Q^2=e^2+g^2$ then the result is similar for the tunneling of uncharged particle from Kerr-Newman blackhole[18].

5.12 : Uncharged particle tunneling from Kerr-Newman-NUT blackhole :

5.12.1 Kerr-Newman-NUT blachkhole: The Kerr-Newman-NUT blackhole metric can be given by [179]

$$ds^2=-\frac{(\Delta-a^2\sin^2\theta)}{\rho^2}dt^2+\frac{\rho^2}{\Delta}dr^2+\rho^2d\theta^2+\frac{\sin^2\theta[r^2+(l+a)^2]^2-\Delta(a\sin^2\theta+4l\sin^2\frac{\theta}{2})^2}{\rho^2}d\phi$$

$$+\frac{2[\Delta(a\sin^2\theta+4l\sin^2\frac{\theta}{2})-a\sin^2\theta\,[r^2+(l+a)^2]}{\rho^2}dtd\phi \qquad \text{..........................(5.12.1)}$$

where $\Delta=r^2+a^2+e^2+g^2-l^2-2Mr,\qquad \rho^2=r^2+(l+a\cos\theta)^2$. Here M is the mass of the blackhole, e and g are the electric and magnetic charges respectively, a is the angular momentum per unit mass, l is the NUT parameter. The event horizon equations are given by $\Delta=0$ which gives

$$r_\pm=M\pm\sqrt{M^2-a^2-e^2-g^2+l^2} \qquad \text{..........................(5.12.2)}$$

The event horizon area of this blackhole is given by

$$A=4\pi(r_+^2+a^2+l^2) \qquad \text{..........................(5.12.3)}$$

and Bekenstein-Hawking entropy

$$S_{BH}=\frac{A}{4}=\pi(r_+^2+a^2+l^2)=\pi[2M^2+2M\sqrt{M^2-a^2-e^2-g^2+l^2}-e^2-g^2+2l^2] \qquad \text{..........................(5.12.4)}$$

5.12.2 Dragging coordinate system and infinite red- shift surface of Kerr-Newman-NUT blachkhole :

The infinite red shift surface is given by $g_{00} = 0$ which gives

$$r_{\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta - e^2 - g^2 + l^2} \quad \ldots\ldots(5.12.5)$$

Obviously the infinite red shift surface does not coincide with the event horizon surface , which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Also there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (5.12.1) into a dragging coordinate system.

Let $\Omega = \dfrac{d\phi}{dt} = -\dfrac{g_{03}}{g_{00}}$ ……………………(5.12.6)

where Ω is the angular velocity.

For the metric (5.12.1) we have,

$$g_{00} = \frac{-(\Delta - a^2\sin^2\theta)}{\rho^2},\quad g_{11} = \frac{\rho^2}{\Delta},\quad g_{22} = \rho^2,$$

$$g_{33} = \frac{\sin^2\theta[(r^2+(l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}{\rho^2}$$

$$g_{03} = \frac{\Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}) - a\sin^2\theta[(r^2+(l+a)^2]}{\rho^2}$$

From (5.12.6), $$\Omega = \frac{d\phi}{dt} = -\frac{\Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}) - a\sin^2\theta[r^2+(l+a)^2]}{\sin^2\theta[(r^2+(l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}$$

……………………………(5.12.7)

At the horizon the angular velocity becomes,

$$\Omega_h = \frac{a}{r_+^2 + (l+a)^2} \quad \ldots\ldots(5.12.8)$$

The line element (5.12.1) in the dragging coordinate system becomes,

$$ds^2 = \hat{g}_{00}\,dt^2 + g_{11}dr^2 + g_{22}d\theta^2 \quad \ldots\ldots(5.12.9)$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2 \sin^2\theta}{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}$.

The line element (5.12.9) represents a 3-dimensional hypersurface of 4-dimensional spacetime. The infinite red-shift surface now coincide with the event horizon surface in the dragging coordinate system. So the geometrical optical limit can be applied now.

5.12.3 Painleve-like coordinate transformation and null geodesics of Kerr-Newman-NUT blachkhole:

To investigate the Hawking radiation as tunneling process it is necessary to eliminate coordinate singularity at the event horizon. In the expression (5.12.9) , there still exists coordinate singularity at the event horizon in the dragging coordinate system. So we continue performing a general Painleve coordinate transformation[173]. This transformation can done by

$$dt_{K-N-NUT} = dt + F(r,\theta)dr + G(r,\theta)d\theta \qquad \text{.............................(5.12.10)}$$

where $F(r,\theta)$ and $G(r,\theta)$ are two determined functions of r and θ , and satisfy the integrability condition ,

$$\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r} \qquad \text{........................(5.12.11)}$$

Thus from (5.12.9) we obtain,

$$ds^2 = \hat{g}_{00}\,dt^2 + \left\{\hat{g}_{00}\,F^2(r,\theta) + g_{11}\right\}dr^2 + \left\{\hat{g}_{00}\,G^2(r,\theta) + g_{22}\right\}d\theta^2 + 2G(r,\theta)\,\hat{g}_{00}\,dtd\theta$$

$$+2F(r,\theta)G(r,\theta)\,\hat{g}_{00}\,drd\theta + 2F(r,\theta)\,\hat{g}_{00}\,dtdr$$

$$\text{...............................(5.12.12)}$$

We demand that constant time- slices are flat Euclidean space in radial. So we set,

$$\hat{g}_{00}\,F^2(r,\theta) + g_{11} = 1$$

$$\Rightarrow F(r,\theta) = \pm\sqrt{\frac{1-g_{11}}{\hat{g}_{00}}} \qquad \text{..............................(5.12.13)}$$

From equation (5.12.11) , $G(r,\theta) = \int \frac{\delta F(r,\theta)}{\delta\theta}dr + C(\theta)$ ....(5.12.14)

where $C(\theta)$ is an arbitrary analytic function of θ .

Substituting the value of $F(r,\theta)$ into equation (5.12.12) we get,

$$ds^2 = \hat{g}_{00}\, dt^2 + dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2\sqrt{\hat{g}_{00}(1-g_{11})}\; G(r,\theta) dr d\theta$$

$$+ 2\,\hat{g}_{00}\, G(r,\theta)\, dt d\theta \pm 2\sqrt{\hat{g}_{00}(1-g_{11})}\;\; dt dr \qquad \text{.....................(5.12.15)}$$

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon.

According to Landau's theory of the coordinate clock synchronization[174] in a spacetime decomposed in 3+1 dimension, the difference of coordinate times of two events taking place simultaneously in different place is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}} dx^i \qquad (i = 1,2,3) \qquad \text{........................(5.12.16)}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy

$$\frac{\delta}{\delta x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\delta}{\delta x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad , \;(i,j = 1,2,3) \qquad \text{.......................(5.12.17)}$$

Now the metric (5.12.15) in the new coordinate system , has a number of attractive features : (1) the metric is well-behaved at the event horizon; (2) the time coordinate t represents the local proper time for radially free-falling observers; (3) the hypersurfaces of constant time-slices are just flat Euclidean space in the oblate spheroidal coordinates; (4) by substituting the components of the metric (5.12.15) into equation (5.12.17), we see that the metric satisfy the Landau's condition of the coordinate clock synchronization $\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$; (5) the infinite red-shift surface coincide with the event horizon surface so that the WKB approximation can be used. These attractive features are very advantageous for us to discuss Hawking radiation as tunneling and to do an explicit computation of the tunneling probability at the event horizon.

Now in order to investigate the tunneling process we evaluated the radial null geodesics described by equation (5.12.15) . Since the tunneling processes take place near the event horizon, so we may consider a particle tunneling from the event horizon as an ellipsoid shell . To conserve the symmetry of the spacetime , we think the particle should be still an ellipsoid shell during the tunneling process i.e. the particle does not have motion in

θ-direction[18]. Under these condition we obtain the radial null geodesics from equation (5.12.15) $(ds^2 = 0 = d\theta^2)$ as

$$\dot{r} = \frac{\sin\theta[\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta)}]}{\sqrt{\sin^2\theta[r^2 + \{l+a\}^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}}$$

.......................(5.12.18)

where a dot denotes differentiation with respect to t and the positive sign $(+)$ represents an outgoing geodesics and the negative sign $(-)$ represents an ingoing geodesics.

5.12.4 Tunneling rate of Kerr-Newman-NUT blachkhole :

We adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out and materialize as a real particle escaping classically to infinity. The negative energy particle is absorbed by the blackhole resulting in a decrease in the mass and the angular momentum of the blackhole. We consider the particle as an ellipsoid shell of energy ω and angular momentum ωa. When the particle's self-gravitation is taken into account, then equation (5.12.15) and (5.12.18) should be modified. To ensure the conservation of energy and angular momentum, we fix the total mass and angular momentum of the blackhole and allow the hole mass and angular momentum to fluctuate. When particle tunnels out , the blackhole mass and angular momentum will become $M-\omega$ and $a(M-\omega)$ respectively.

The shell of energy will move along the modified null geodesic in the radial direction

$$\dot{r} = \frac{\sin\theta[\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \bar{\Delta})}]}{\sqrt{\sin^2\theta[r^2 + \{l+a\}^2]^2 - \bar{\Delta}(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}}$$

.......................(5.12.19)

where $\bar{\Delta} = r^2 - 2(M-\omega) + a^2 + e^2 + g^2 - l^2$ is the horizon equation after the emission of the particle with energy ω .

Now the coordinate ϕ does not appear in the dragged Painleve-Gullstrand metric (5.12.15) . So ϕ is an ignorable coordinate in the Lagrangian function

L.. To eliminate this degree of freedom completely, the action should be written as

$$L = \int_{t_{in}}^{t_{out}} (L - P_\phi \dot{\phi})dt \quad \text{.................................(5.12.20)}$$

So the imaginary part of the action is

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} [\int_{(0,0)}^{(P_r, P_\phi)} (\dot{r}\, dP_r^{\prime} - \dot{\phi}\, dP_\phi^{\prime})] \frac{dr}{\dot{r}} \quad \text{..............................(5.12.21)}$$

where P_r and P_ϕ are two canonical momentum conjugate to r and ϕ respectively.

$r_{in} = M + \sqrt{M^2 - a^2 - e^2 - g^2 + l^2}$ and $r_{out} = (M - \omega) + \sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2}$ are the locations of the event horizon before and after a particle tunnels out, they are just inside and outside the barrier through which the particle tunnels.

We now eliminate the momentum in favor of energy by using Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;\phi,P_\phi)} = \frac{d(M-\omega)}{dP_r} \quad \text{..........................(5.12.22)}$$

$$\dot{\phi} = \frac{dH}{dP_\phi}\Big|_{(\phi;r,P_r)} = a\bar{\Omega}\frac{d(M-\omega)}{dP_\phi} \quad \text{....................(5.12.23)}$$

where $dH_{(\phi;r,P_r)} = \bar{\Omega}\, dJ = a\bar{\Omega}\, d(M-\omega)$ represents the energy change of the blackhole because of the loss of the angular momentum when a particle tunnels out, and the dragging angular velocity is given by

$$\Omega = \frac{d\phi}{dt} = -\frac{\bar{\Delta}(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}) - a\sin^2\theta[r^2 + (l+a)^2]}{\sin^2\theta[(r^2 + (l+a)^2]^2 - \bar{\Delta}(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}$$

.....................(5.12.24)

Substituting equations (5.12.19), (5.12.22) (5.12.23) and (5.12.24) into equation (5.12.21) and noting that we must choose the positive sign in equation (5.12.19) as the particle is propagating from inside to outside the horizon, then we have

$$\operatorname{Im} S = \operatorname{Im}[\int_{r_{in}}^{r_{out}} \int_{M}^{M-\omega} (1 - a\Omega^{\prime}) \frac{dr}{\dot{r}} d(M - \omega^{\prime})$$

$$= \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M-\omega} (1 - a\Omega') \frac{\sqrt{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta'(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}}{\sin\theta[\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta')}]} drd(M - \omega')$$

....(5.12.25)

where

$$\bar{\Delta} = r^2 + a^2 + e^2 + g^2 - l^2 - 2(M - \omega')r = (r - r_+^{/})(r - r_-^{/})$$

$$r_{\pm}^{/} = (M - \omega') \pm \sqrt{(M - \omega')^2 - a^2 - e^2 - g^2 + l^2}$$

We multiply and divide the integrand with $\rho^2 + \sqrt{\rho^2(\rho^2 - \Delta')}$ to Obtain,

$\mathrm{Im}\, S =$

$$\mathrm{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M-\omega} (1 - a\Omega_h') \frac{[\rho^2 + \sqrt{\rho^2(\rho^2 - \Delta')}]\sqrt{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta'(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}}{(r - r_+^{/})(r - r_-^{/})\rho^2 \sin\theta} drd(M - \omega')$$

....(5.12.26)

We see that $r = r_+^{/}$ is a pole of order one. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,[178]

$$\int_{r_{in}}^{r_{out}} \frac{[\rho^2 + \sqrt{\rho^2(\rho^2 - \Delta')}\sqrt{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta'(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}}{(r - r_+^{/})(r - r_-^{/})\rho^2 \sin\theta} dr$$

$$= -2\pi i \frac{r_+^{/2} + (a+l)^2}{r_+^{/} - r_-^{/}}$$

............................(5.12.27)

Therefore

$$\mathrm{Im}\, S = -\mathrm{Im} \int_{M}^{M-\omega} (1 - a\Omega_h^{/}) 2\pi i \frac{r_+^{2} + (l+a)^2}{(r_+^{/} - r_-^{/})} d(M - \omega')$$

$$= -2\pi \int_{M}^{M-\omega} \frac{2(M - \omega')^2 + 2(M - \omega')\sqrt{(M - \omega')^2 - a^2 - e^2 - g^2 + l^2} - a^2 - e^2 - g^2 + 2l^2 + 2al}{\sqrt{(M - \omega')^2 - a^2 - e^2 - g^2 + l^2}} d(M$$

Finishing the integral we have

$$\operatorname{Im} S = -\pi[(M-\omega)^2 - M^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2} - M\sqrt{M^2 - a^2 - e^2 - g^2 + l^2}]$$

$$\text{..............................(5.12.28)}$$

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{2\pi[(M-\omega)^2 - M^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2} - M\sqrt{M^2 - a^2 - e^2 - g^2 + l^2}]}$$

$$\text{.............................(5.12.29)}$$

Using Bekenstein-Hawking entropy formula $S_{BH} = \pi(r_+^2 + a^2 + l^2)$,

we have $S_{BH}(M) = \pi[2M^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2 + l^2} - e^2 - g^2 + 2l^2]$

$$S_{BH}(M-\omega) = \pi[2(M-\omega)^2 + 2(M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2} - e^2 - g^2 + 2l^2]$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$$
$$= 2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2} - M^2 - M\sqrt{M^2 - a^2 - e^2 - g^2 + l}$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. From equation (5.12.29) we have

$\Gamma \sim e^{\Delta S_{BH}}$ ....(5.12.30)

5.12.5 Concluding remarks:

In this paper, we have presented the Hawking radiation as tunneling from Kerr-Newman-NUT blackhole by applying Krause-Parikh-Wilczek's semi-classical quantum tunneling method[13,14,15,16]. We find that the emission rate at the event horizon is equal to the difference of Bekenstein-Hawking entropy before and after the emission of a particle. The Hawking temperature of this type of blackhole recovered by expanding ΔS_{BH} in $(\omega - \omega_0)$ and neglecting the higher order terms.

According to the reference [178] expression (5.12.30) is just the emission rate of the Hawking radiation which ignores self-gravitation action. So we conclude that the real radiation spectrum of this type of blackhole is not precisely thermal when energy conservation and angular momentum conservation are taken into account. The tunneling rate we obtained is more accurate and is a good correction to Hawking pure thermal spectrum.

In special case , if we put $l=0$ and we assume the equivalent charge $Q^2 = e^2 + g^2$ then the result is similar for the tunneling of uncharged particle from Kerr-Newman blackhole[18]. If $l = e = g = 0$ then the result reduces to the Kerr blackhole[18] .For $l = a = g = 0$ then the result is fit for Reissner-Nordström blackhole and supports the Parikh-Wilczek's result[14]. Also if we assume $a = l = e = g = 0$ then the result is supports for the Schwarzschild blackhole obtained by the Parikh-Wilczek's result[14].

The result we derived above shows that the blackhole radiation causes the spacetime background geometry to be varied. Because of the self-gravitation and energy conservation and angular momentum conservation, the event horizon of blackhole varies with blackhole radiation, namely when the particle outgoes the event horizon will contract and the two turning points pre-contraction and post-contraction are the two points of barrier. The tunneling rate of particle is relevant to the mass M, the angular momentum a,the electric charge e, the magnetic charge g, and the NUT parameters l of the blackhole and satisfies the underlying unitary theory.

CHAPTER SIX

HAKING RADIATION AS TUNNELING OF CHARGED PARTICLE

6.1 Charged particle tunneling: Parikh and Wilczek [14] calculate the emission rate of the uncharged massless particles by the tunneling process from the event horizon of the Schwarzschild and R-N blackholes. In this section, we investigate the tunneling behavior of charged massive particle and calculate its emission rate from a charged R-N blackhole. To calculate the tunneling rate of charged massive particle, one must overcome two additional difficulties. The first is that one has to decide the equation of motion of a charged massive particle, since the radial null geodesics is only applicable to describe the tunneling behavior of uncharged massless particle from the event horizon. The trajectory followed by a charged massive particle is not light-like but subject to Lorentz forces, different from the null geodesics of an uncharged mass less particle. Here, to investigate the tunneling behavior of charged massive particle, we consider the phase velocity and group velocity of the de Broglie wave corresponding to the outgoing particle. The second difficulty is how to take into account the effect of the electro-magnetic field when the charged massive particle tunnels out from the event horizon. One must take into account not only the conservation of energy but also the electric charge conservation. Here , we adopt a slightly modified tunneling picture, that is we consider the charged massive particle as a charged conducting ellipsoid shell carrying energy ω and electric charge q. To take into account for the effect of the electro-magnetic field , we consider a matter-gravity system that consists of the blackhole and electro-magnetic field outside the hole. Taking into account the particle's self-gravitation, the conservation of energy and electric charge, we must fix the total mass and total electric charge of the spacetime but allow those of the blackhole to vary also.

6.2 Charged particle tunneling from R-N blackhole:

6.2.1 Painleve- like coordinate transformation of R-N blackhole:

The line element of Reissner-Nordstr$o$ m blackhole is given by,

$$ds^2 = -(1-\frac{2M}{r}+\frac{Q^2}{r^2})dt^2 + (1-\frac{2M}{r}+\frac{Q^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad \text{..................(6.2.1)}$$

To investigate the Hawking radiation as tunneling process we should adopt Painleve-like coordinate transformation.

According to the transformation

$$t_{R-N} = t - \int\sqrt{\frac{1-g(r)}{f(r)g(r)}}dr \quad \text{..........................(6.2.2)}$$

The metric (6.2.1) reduces to,

$$ds^2 = -f(r)dt^2 + 2\sqrt{f(r)}\sqrt{\frac{1}{g(r)}-1}\ dtdr + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

$$\text{.............(6.2.3)}$$

where $f(r) = g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

The metric(6.2.3) can be rewritten as,

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\ dtdr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\text{..........(6.2.4)}$$

6.2.2 Phase velocity and electro-magnetic potential of R-N blackhole:

The line element of R-N blackhole and the null geodesics in Painleve-like coordinate transformation are given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\ dtdr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\text{..........(6.2.5)}$$

The radial null geodesics are given by,

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} \quad \text{..(6.2.6)}$$

where the positive sign (+) represents the outgoing geodesics and the negative sign (-) represents the ingoing geodesics under the implicit assumption that t increases towards the future.

The elctro-magnetic potential is given by,

$$A_\mu = (A_t, 0, 0, 0) \quad \text{...........................(6.2.7)}$$

where $A_t = -\frac{Q}{r}$

Since the charged massive quanta does not follow the radial null geodesics (6.2.6), so we consider the outgoing particle is a massive shell (de Broglie s-wave). According to de Broglie hypothesis , this massive shell is a sort of de Broglie s-wave. The approximation wave equation is given by [176],

$$\psi(r,t) = Ce^{i(\int_{r_i-\varepsilon}^{r} P_r dr - \omega t)} \quad \text{..............................(6.2.8)}$$

where $r_i - \varepsilon$ is the initial location of the particle.

If we let $\int_{r_i-\varepsilon}^{r} P_r dr - \omega t = \phi_0$ ....(6.2.9)

then we have $\frac{dr}{dt} = \dot{r} = \frac{\omega}{k}$ ....(6.2.10)

where k is the de Broglie wave number.

Comparing equation (6.2.9) with the definition of the phase velocity we know that $\dot{r}$ is the phase velocity of the de Broglie wave. Unlike the electromagnetic wave , the phase velocity v_p of the de Broglie wave is not equal to the group velocity v_g. The definition and relationship between them are,

$$v_p = \frac{dr}{dt} = \dot{r} = \frac{\omega}{k} \quad \text{...............................(6.2.11)}$$

$$v_g = \frac{dr_c}{dt} = \frac{d\omega}{dt} \quad \text{...............................(6.2.12)}$$

$$v_p = \frac{1}{2} v_g \quad \text{...............................(6.2.13)}$$

Since the tunneling across the barrier is an instantaneous process, there are two simultaneous events during the process , one is particle tunneling into the barrier and another is particle tunneling out the barrier. According to Landau's theory of the coordinate clock synchronization[174], the difference of coordinate times of these two simultaneous events is

$$dt = -\frac{g_{0i}}{g_{00}} dx^i = -\frac{g_{01}}{g_{00}} dr_c \quad (d\theta = d\phi = 0) \quad \text{...............................(6.2.14)}$$

where r_c denote the location of the tunneling particle. The group velocity is

$$v_g = \frac{dr_c}{dt} = -\frac{g_{00}}{g_{01}} \quad \text{...............................(6.2.15)}$$

and the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2} v_g = -\frac{1}{2}\frac{g_{00}}{g_{01}} = \frac{1}{2}\frac{(r^2 - 2Mr + Q^2)}{r\sqrt{2Mr - Q^2}} \quad \text{......................(6.2.16)}$$

6.2.3 Tunneling rate of charged particle from R-N blackhole :

We consider the spacetime as dynamical and incorporating the self-gravitation effect of the tunneling particle when the energy conservation and the electric charge conservation are taken into account. We assume that the total ADM mass and charge of the hole-particle system are held fixed whereas the mass and the charge of the hole are allowed to fluctuate, then the mass and the charge will become $M-\omega$ and $Q-q$ when a particle with energy ω and charge q has tunneled from the event horizon. So considering the charged massive particle tunnel's out from the event horizon along the radial direction, we should modify the metric (6.2.5) as ,

$$ds^2 = -\{ 1-\frac{2(M-\omega)}{r}+\frac{(Q-q)^2}{r^2})dt^2+2\sqrt{\frac{2(M-\omega)}{r}-\frac{(Q-q)^2}{r^2}}\,dtdr+dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

......................(6.2.17)

and the radial geodesics are given by,

$$\dot{r}=\frac{1}{2}\;\frac{r^2-2(M-\omega)r+(Q-q)^2}{r\sqrt{2(M-\omega)r-(Q-q)^2}} \qquad \text{....................(6.2.18)}$$

Also the non-zero component of electro-magnetic potential becomes

$$A_t=-\frac{Q-q}{r} \qquad \text{....................................(6.2.19)}$$

As the Lagrangian function of the electro-magnetic field corresponding to the generalized coordinates described by A_μ is $\frac{-1}{4}F_{\mu\nu}F^{\mu\nu}$, we can find that the generalized coordinate $A_\mu=(A_t,0,0,0)$ is an ignorable coordinate.

In order to eliminate the degree of freedom corresponding to A_μ, the imaginary part of the action for the charged massive particle should be written as,

$$\operatorname{Im} S=\operatorname{Im}\int_{t_i}^{t_f}(L-P_{A_t}\dot{A}_t)dt=\operatorname{Im}\int_{r_{in}}^{r_{out}}(P_r-\frac{P_{A_t}\dot{A}_t}{\dot{r}})dr=\operatorname{Im}[\int_{r_{in}}^{r_{out}}\left\{\int_{(0,0)}^{(P_r,P_{A_t})}(dP_r{}'-\frac{\dot{A}_t\,dP_{A_t}{}'}{\dot{r}})\right\}dr]$$

............................(6.2.20)

where P_{A_t} is the electro-magnetic field's canonical momentum conjugate to A_t .

Applying Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;A_t,P_{A_t})} \quad \text{..............................(6.2.21)}$$

$$\dot{A}_t = \frac{dH}{dP_{A_t}}\Big|_{(A_t;r,P_r)} \quad \text{..............................(6.2.22)}$$

Substituting equation (6.2.21) and (6.2.22) into equation (6.2.20) we obtain,

$$\operatorname{Im} S = \operatorname{Im}[\{\int_{r_{in}}^{r_{out}}\left\{\int_{(M,E_Q)}^{(M-\omega,E_{Q-q})}(\frac{1}{\dot{r}}(dH)_{r;A_t,P_{A_t}} - \frac{1}{\dot{r}}(dH)_{A_t;r,P_r})\right\}]dr$$

...................(6.2.23)

where

$(dH)_{r;A_t,P_{A_t}} = d(M-\omega') = -d\omega'$.

$(dH)_{A_t;r,P_r} = -\dfrac{Q-q'}{r}dq'$.

Putting the value of $\dot{r}$ from (6.2.18) into (6.2.23) we get,

$$\operatorname{Im} S = -\operatorname{Im}[\int_{r_{in}}^{r_{out}}\int_{(0,0)}^{(\omega,q)}\left\{\frac{2r\sqrt{2(M-\omega')r-(Q-q')^2}}{r^2-2(M-\omega')r+(Q-q')^2}d\omega' - \frac{2(Q-q')\sqrt{2(M-\omega')r-(Q-q')^2}}{r^2-2(M-\omega')r+(Q-q')^2}dq'\right\}dr]$$

$$= -\operatorname{Im}[\int_{r_{in}}^{r_{out}}\int_{(0,0)}^{(\omega,q)}\left\{\frac{2r\sqrt{2(M-\omega')r-(Q-q')^2}}{(r-r_+')(r-r_-')}d\omega' - \frac{2(Q-q')\sqrt{2(M-\omega')r-(Q-q')^2}}{(r-r_+')(r-r_-')}dq'\right\}dr]$$

..................................(6.2.24)

where

$$r_\pm' = (M-\omega') \pm \sqrt{(M-\omega')^2-(Q-q')^2} \quad \text{...........................(6.2.25)}$$

$$r_{in} = M + \sqrt{M^2-Q^2} \quad \text{..............................(6.2.26)}$$

$$r_{out} = (M-\omega) + \sqrt{(M-\omega)^2-(Q-q)^2} \quad \text{................................(6.2.27)}$$

We see that $r = r_+'$ is a pole of order one. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\operatorname{Im} S = -2\pi\int_{(0,0)}^{(\omega,q)}[\frac{r_+'^2}{(r_+'-r_-')}d\omega' - \frac{r_+'(Q-q')}{(r_+'-r_-')}dq'] \quad \text{....................(6.2.28)}$$

Finishing the integral we have,

$$\operatorname{Im} S = -\frac{\pi}{2}[\{(M-\omega)+\sqrt{(M-\omega)^2-(Q-q)^2}\}^2 - \{M+\sqrt{M^2-Q^2}\}^2] = -\frac{1}{2}\Delta S_{BH}$$

..................................(.6.2.29)

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission.

The tunneling rate is therefore

$\Gamma \sim e^{-2\operatorname{Im} S} = e^{\Delta S_{BH}}$ ....(6.2.30)

Now , if we expand ΔS_{BH} in terms of ω, q and take only the first order term then

$\Delta S_{BH} = -\beta(\omega - \omega_0)$ where $\beta = \frac{2\pi[M+\sqrt{M^2-Q^2}]^2}{\sqrt{M^2-Q^2}}$ is the inverse Hawking temperature and $\omega_0 = \frac{Qq}{M+\sqrt{M^2-Q^2}}$.

So from equation (6.2.30) we obtain,

$\Gamma \sim e^{-\beta(\omega-\omega_0)}$ ....(6.2.31)

From equation (6.2.31) we see that the corrected spectrum is not precisely thermal. Only the leading order term gives the thermal Boltzmann factor $e^{-\beta(\omega-\omega_0)}$.

6.3 Charged Particle Tunneling from Kerr Blackhole:

6.3.1 Painleve- like coordinate transformation of Kerr Blackhole: The behavior of a scalar field theory near the event horizon in a rotating blackhole background can be effectively described by a two dimensional field theory in a gauge field background. Based upon this concept Tao Zhu [181] proposed that the quantum tunneling from rotating blackhole can be treated as "charged particle's "tunneling process in its effectively two dimensional metric. Using this view point and considering the corresponding 'gauge charge' conservation he calculate the non-thermal tunneling rate from Kerr blackhole and his results are consistent with the Parikh-Wilczek's original result for spherically symmetric blackholes.

The line element of Kerr metric in the Boyer-Lindquist coordinate system is given by,

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - a\,dt]^2$$

....................................(6.3.1)

where

$$\Delta = r^2 - 2Mr + a^2$$
$$\rho^2 = r^2 + a^2 \cos^2\theta$$
$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum and r is the radial distance from the center of the body. The equation of the event horizon is given by,

$\Delta = 0$ which gives, $r_\pm = M \pm \sqrt{M^2 - a^2}$, $M^2 > a^2$.

Now apply the technique of the dimensional reduction near the horizon of Kerr blackhole in 4-dimensional behaves as the two dimensional spherically symmetric line element in the region near the horizon is given by [181]

$$ds_2^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 \quad \text{.......................................(6.3.2)}$$

Where

$f^{-1}(r) = g_{11}$, $g_{01} = g_{10} = 0$, $g_{00} = -f(r)$, $\Phi = r^2 + a^2$, dilaton and gauge charge field A_μ is given by

$$A_t = -\frac{a}{r^2 + a^2}, \quad A_r = 0 \quad \text{.......................................(6.3.3)}$$

Now since the tunneling method deals with the region very close to the horizon, one can investigate the quantum tunneling effect of Kerr blackhole by using this two dimensional metric [182].

We use the Painleve-like coordinate transformation as

$$dt_k = dt + \frac{\sqrt{2Mr}}{1 - \frac{2Mr}{r^2 + a^2}}dr \quad \text{....................................(6.3.4)}$$

Then the line element becomes

$$ds^2 = -(1 - \frac{2Mr}{r^2 + a^2})dt_k^2 + 2\sqrt{\frac{2Mr}{r^2 + a^2}}dt_k dr + dr^2 \quad \text{................................(6.3.5)}$$

6.3.2 Tunneling rate of charged particle from Kerr blackhole: The radial null geodesics are given by ($ds^2 = 0$)

$$\dot{r} = \pm 1 - \sqrt{\frac{2Mr}{r^2 + a^2}} \quad \text{..(6.3.6)}$$

Where the upper(lower) sign corresponding to the outgoing(ingoing) geodesics.

As we considering a rotating Kerr blackhole , so the rotation degree of freedom should be well addressed also. So energy conservation and angular momentum conservation should be taken into account. Now from the action [183]

$$I = \int dtdr(r^2 + a^2)\phi^*_{lm}[\frac{r^2+a^2}{\Delta}(\delta_t + \frac{iam}{r^2+a^2})^2 - \delta_r \frac{\Delta}{r^2+a^2}\delta_r]\phi_{lm} \quad \text{...............(6.3.7)}$$

We see that only the magnetic quantum number m of particle is relevant near the horizon and the particle which contains quantum number m behaves as a "charged " particle with gauge charge m in the background gauge field A_μ. In this sense the angular momentum conservation means gauge charge m conservation. So we can treat the tunneling process as the "charged" particle's tunneling. When self-gravitation of the tunneling particle is included then equation (6.3.6) should be modified by $M \to M - \omega,\ a = \frac{J}{M} \to a' = \frac{J-m}{M-\omega}$, thus

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M-\omega)r}{r^2 + a'^2}} \quad \text{..(6.3.8)}$$

where ω is the particle energy and J is the total angular momentum of the Kerr blackhole.

When we investigate a charged particle's tunneling process , the effect of the gauge field should be taken into account . For this reason, we write the Lagrangian function of the system as $L = L_A + L_m$ where $L_A = -\frac{F_{\mu\nu}F^{\mu\nu}}{4}$ is the Lagrangian function of the gauge field corresponding to the generalized coordinates $A_\mu = (-\frac{a}{r^2+a^2},\ 0)$. When a charged particle tunnels out , the system transit from one state to another. From the expression of L_A we find that A_t is an ignorable coordinate. To eliminate the freedom corresponding to A_t the action should be written as

$$I = \int_{t_i}^{t_f} (L - P_{A_t} \dot{A}_t)dt \quad \text{................................(6.3.9)}$$

The emission rate of the tunneling particle is related by

$$\Gamma \sim e^{-2\,\mathrm{Im}\,I} \quad \text{..................................(6.3.10)}$$

The imaginary part of the action is

$$\mathrm{Im}\,I = \mathrm{Im}\int_{r_i}^{r_f}(P_r - \frac{P_{A_t}\dot{A}_t}{\dot{r}})dr$$

$$= \mathrm{Im}\int[dP'_r - \frac{\dot{A}_t}{\dot{r}}dP'_{A_t}]dr \quad \text{...(6.3.11)}$$

where P_{A_t} is the gauge fields canonical momentum conjugate to A_t.

By applying Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;A_t,P_{A_t})} = \frac{d(M-\omega')}{dP_r}$$
$$\dot{A} = \frac{dH}{dP_{A_t}}\Big|_{(A_t;r,P_r)} = -\frac{a'}{r^2+a'^2}\frac{dm'}{dP_{A_t}} \quad \text{..............................(6.3.12)}$$

where $a' = \frac{J-m'}{M-\omega'}$ and $dH_{(A_t;r,P_r)}$ represents the energy change of the blackhole because of the loss of the gauge charge m when a particle tunnels out.

As the particle propagating from inside to outside the event horizon, so we take positive sign of equation (6.3.8).

So we have

$$\operatorname{Im} I = \operatorname{Im}\int_{(M,0)}^{(M-\omega,m)}\int_{r_{in}}^{r_{out}}\left[d(M-\omega') + \frac{a'}{r'^2+a'^2}dm'\right]\frac{dr}{1-\sqrt{\frac{2(M-\omega')r}{r^2+a'^2}}}$$

..................(6.3.13)

where $r = r_+' = (M-\omega') + \sqrt{(M-\omega')^2 - a'^2}$ is a simple pole in the above equation . The integral can be evaluated by deforming the contour around the pole. In this way we obtain

$$\operatorname{Im} I = -2\pi\int_{(M,J)}^{(M-\omega,J-m)}\frac{r_+'^2+a'^2}{r_+'-r_-'}\left[d(M-\omega') - \frac{a'}{r_+'^2+a'^2}d(J-m')\right] \quad \text{.............(6.3.14)}$$

where $r_- = (M-\omega') - \sqrt{(M-\omega')^2 - a'^2}$ .and $dm' = -d(J-m')$.

Now the Hawking temperature on the event horizon of the Kerr blackhole is given by

$$T' = \frac{r_+' - r_-'}{4\pi(r_+'^2 + a'^2)} \quad \text{...(6.3.15)}$$

Thus we have

$$\operatorname{Im} I = -\frac{1}{2}\int_{(M,J)}^{(M-\omega,J-m)}\frac{1}{T'}\left[d(M-\omega') - \frac{a'}{r_+'^2+a'^2}d(J-m')\right] = -\frac{1}{2}\Delta S_{BH} \,.$$

..................................(6.3.16)

where ΔS_{BH} is the difference of the entropies of the blackhole before and after the emission . The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} I} = e^{\Delta S_{BH}} \quad \text{................................(6.3.17)}$$

6.4 Charged particle tunneling from Kerr-Newmann Blackhole:

6.4.1 Phase velocity and electro-magnetic potential of Kerr-Newman Blackhole: The line element of Kerr-Newman metric in the Boyer-Lindquist coordinate system is given by,

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a\,dt]^2 \quad \text{..........................(6.4.1)}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum , Q is the electric charge. The equation of the event horizon is given by,

$\Delta = 0$ which gives, $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$, $M^2 > a^2 + Q^2$.

and the 4-dimensional electro-magnetic potential is given by[21]

$$A_a = -\rho^{-2}Qr[(dt)_a - a\sin^2\theta\,(d\phi)_a] \quad \text{........................(6.4.2)}$$

The line element of Kerr-Newman blackhole in dragged Painleve-Gullstrand coordinate is already given in (5.9.11). Since

$$\left(\frac{\delta}{\delta t_d}\right)^a = \left(\frac{\delta}{\delta t_{KN}}\right)^a + \Omega\left(\frac{\delta}{\delta\phi}\right)^a \quad \text{............................(6.4.3)}$$

we can easily obtain the component of the electromagnetic potential in the dragging coordinate system

$$A_t = A_a\left(\frac{\delta}{\delta t_d}\right)^a = -\rho^{-2}Qr[1 - a\Omega\sin^2\theta],\ A_r = A_\theta = 0 \quad \text{.....................(6.4.4)}$$

where $\Omega = -\frac{g_{03}}{g_{33}}$ is the dragged angular velocity and in the Painleve-Gullstrand coordinate transformation the component of the electro-magnetic potential is unchanged $A_t = -\rho^{-2}Qr[1 - a\Omega\sin^2\theta],\quad A_r = A_\theta = 0$

.....................(6.4.5)

Since the charged massive quanta does not follow the radial null geodesics , so we consider the outgoing particle is a massive shell (de Broglie s-wave). According to de Broglie hypothesis , this massive shell is a sort of de Broglie s-wave. The approximation wave equation is given by [176],

$$\psi(r,t) = C e^{i\left(\int_{r_i-\varepsilon}^{r} P_r dr - \omega t\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4.6)$$

where $r_i - \varepsilon$ is the initial location of the particle.

If we let $\int_{r_i-\varepsilon}^{r} P_r dr - \omega t = \phi_0$ ....(6.4.7)

then we have $\frac{dr}{dt} = \dot{r} = \frac{\omega}{k}$ ....(6.4.8)

where k is the de Broglie wave number.

Comparing equation (6.4.7) with the definition of the phase velocity we know that $\dot{r}$ is the phase velocity of the de Broglie wave. Unlike the electromagnetic wave , the phase velocity v_p of the de Broglie wave is not equal to the group velocity v_g. The definition and relationship between them are,

$$v_p = \frac{dr}{dt} = \dot{r} = \frac{\omega}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4.9)$$

$$v_g = \frac{dr_c}{dt} = \frac{d\omega}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4.10)$$

$$v_p = \frac{1}{2} v_g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4.11)$$

Since the tunneling across the barrier is an instantaneous process, there are two simultaneous events during the process , one is particle tunneling into the barrier and another is particle tunneling out the barrier. According to Landau's theory of the coordinate clock synchronization[174], the difference of coordinate times of these two simultaneous events is

$$dt = -\frac{\hat{g}_{01}}{\hat{g}_{00}} dx' = -\frac{\hat{g}_{01}}{\hat{g}_{00}} dr_c \qquad (d\theta = d\phi = 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4.12)$$

where r_c denote the location of the tunneling particle. The group velocity is

$$v_g = \frac{dr_c}{dt} = -\frac{\hat{g}_{00}}{\hat{g}_{01}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4.13)$$

and the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2} v_g = -\frac{1}{2}\frac{\hat{g}_{00}}{\hat{g}_{01}} = -\frac{1}{2}\frac{\hat{g}_{00}}{\sqrt{\hat{g}_{00}(1-g_{11})}} = \frac{\Delta\rho}{2\sqrt{(\rho^2-\Delta)[(r^2+a^2)^2-\Delta a^2\sin^2\theta]}}$$

..............................(6.4.14)

6.4.2 Tunneling rate of Charged particle from Kerr-Newmann Blackhole:

We consider the spacetime as dynamical and incorporating the self-gravitation effect of the tunneling particle when the energy conservation, angular momentum conservation and the electric charge conservation are taken into account. We assume that the total ADM mass ,angular momentum and charge of the hole-particle system are held fixed whereas the mass ,angular momentum and the charge of the hole are allowed to fluctuate, then the mass and the charge will become $M-\omega$ and $Q-q$ when a particle with energy ω and charge q has tunneled from the event horizon. So considering the charged massive particle tunnel's out from the event horizon along the radial direction, we should modify the equation (6.4.14) as ,

$$\dot{r} = \frac{\bar{\Delta}\rho}{2\sqrt{(\rho^2-\bar{\Delta})[(r^2+a^2)^2-\bar{\Delta}a^2\sin^2\theta]}} \quad ....(6.4.15)$$

where $\bar{\Delta} = r^2 + a^2 + (Q-q)^2 - 2(M-\omega)r$ is the horizon equation after the emission of the particle with energy ω and charge q. we taken into account the effect of the electro-magnetic field to investigate the tunneling of charged particle. That is , the matter- gravity system consists of the blackhole and the electro-magnetic field outside the hole. We write the Lagrangian function of the matter-gravity system as

$$L = L_m + L_e \quad ....(6.4.16)$$

where $L_e = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian function of the electro-magnetic field corresponding to the generalized coordinates described by $A_\mu = (A_t,0,0)$ in the dragged Painleve-Gullstrand – Kerr-Newman coordinate system [184]. When a charged particle tunnels out , the system transit from one state to another. But from the expression of L_e ,we find that $A_\mu = (A_t,0,0)$ is an ignorable coordinate. In addition , the coordinate ϕ does not appear in the line element expressions (5.9.5) and (5.9.11)[Ref. Chapter Five]. In order to eliminate these two degrees of freedom corresponding to A_μ completely, the action for the classically forbidden trajectory should be written as

$$S = \int_{t_{in}}^{t_{out}} (L - P_{A_t}\dot{A}_t - P_\phi \dot{\phi})dt \quad ....(6.4.17)$$

which is related to the emission rate of the tunneling particle by

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} \quad ....(6.4.18)$$

The imaginary part of the action for the charged massive particle is

$$\operatorname{Im} S = \operatorname{Im}\left\{ \int_{r_{in}}^{r_{out}} \left(P_r - \frac{P_{A_t}\dot{A}_t}{\dot{r}} - \frac{P_\phi \dot{\phi}}{\dot{r}}\right) dr \right\}$$

$$= \operatorname{Im}\left\{ \int_{r_{in}}^{r_{out}} \left[\int_{(0,0,0)}^{(P_r, P_{A_t}, P_\phi)} (\dot{r}\, dp'_r - \dot{A}_t\, dP'_{A_t} - \dot{\phi}\, dP'_\phi) \right] \frac{dr}{\dot{r}} \right\} \quad \text{.....................(6.4.19)}$$

where P_r, P_{A_t} and P_ϕ are the canonical momentum conjugate to r, A_t and ϕ respectively. Also $r_{in} = M + \sqrt{M^2 - Q^2 - a^2}$, $r_{out} = (M-\omega) + \sqrt{(M-\omega)^2 - (Q-q)^2 - a^2}$ are the locations of the event horizons before and after the charged particle emission. According to the Hamilton's equations, we have

$$\dot{r} = \frac{dH}{dP_r}\Big|_{(r;\phi,P_\phi;A_t,P_{A_t})} = \frac{d(M-\omega)}{dP_r} \quad \text{.................................(6.4.20)}$$

$$\dot{\phi} = \frac{dH}{dP_\phi}\Big|_{(\phi;r,P_r;A_t,P_{A_t})} = a\Omega \frac{d(M-\omega)}{dP_\phi} \quad \text{.............................(6.4.21)}$$

$$\dot{A}_t = \frac{dH}{dP_{A_t}}\Big|_{(A_t;r,P_r;\phi,P_\phi)} = \Phi \frac{d(Q-q)}{dP_{A_t}} \quad \text{............................(6.4.22)}$$

where the dragging angular velocity and the electro-magnetic potential in the dragging coordinate system are given by

$$\Omega = \frac{a(r^2 + a^2 - \Delta)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \quad \text{..............................(6.4.23)}$$

$$\Phi = -\frac{(Q-q)r(r^2+a^2)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \quad \text{.............................(6.4.24)}$$

Substituting equation(6.4.15) and (6.4.20)-(6.4.24) into equation (6.4.19) the imaginary part of the action becomes

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_{(M,Q)}^{(M-\omega,Q-q)} [d(M-\omega') - a\Omega' d(M-\omega') - \Phi' d(Q-q')] \frac{dr}{\dot{r}}$$

$$= \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_{(M,Q)}^{(M-\omega,Q-q)} [(1 - a\Omega')d(M-\omega') - \Phi' d(Q-q')] \frac{2\sqrt{(\rho^2 - \Delta')[(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta]}}{\Delta' \rho}\, dr$$

$$= \text{Im} \int_{r_{in}}^{r_{out}} \int_{(M,Q)}^{(M-\omega,Q-q)} [\{ 1 - \frac{a^2(r^2+a^2-\Delta')}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} \} \frac{2\sqrt{(\rho^2-\Delta')[(r^2+a^2)^2-\Delta' a^2\sin^2\theta]}}{\Delta'\rho} d(M-\omega')$$

$$- \frac{(Q-q')r(r^2+a^2)}{(r^2+a^2)^2-\Delta' a^2\sin^2\theta} \frac{2\sqrt{(\rho^2-\Delta')[(r^2+a^2)^2-\Delta' a^2\sin^2\theta]}}{\Delta'\rho} d(Q-q')]$$

....(6.4.25)

where

$\Delta' = r^2 + a \quad -(M-\omega')r + (Q-q')^2 = (r - r_+^{/})(r - r_-^{/})$

$r_\pm^{/} = (M-\omega') \pm \sqrt{(M-\omega')^2 - (Q-q')^2 - a^2}$

We see that $r = r_+^{'}$ is a simple pole at the event horizon. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\text{Im}\, S = -\pi \int_{(M,Q)}^{(M-\omega,Q-q)} [\frac{2r_+^{/2}}{(r_+^{/} - r_-^{/})} d(M-\omega') - \frac{2r_+^{/}(Q-q')}{(r_+^{/}-r_-^{/})} d(Q-q')]$$

$$= -\pi \int_{(M,Q)}^{(M-\omega,Q-q)} \frac{r_+^{/}}{r_+^{/} - r_-^{/}} [2r_+^{/} d(M-\omega') - 2(Q-q')d(Q-q')] \quad \text{.................(6.4.26)}$$

Now from $r_+^{/} = (M-\omega') + \sqrt{(M-\omega')^2 - (Q-q')^2 - a^2}$ we obtain the identity

$(r_+^{/} - r_-^{/})dr_+^{/} = 2r_+^{/} d(M-\omega') - 2(Q-q')d(Q-q')$ ....(6.4.27)

Using this identity into equation (6.4.26) we can easily finish the integration and yields a simple expression

$$\text{Im}\, S = -\pi \int_{r_{in}}^{r_{out}} r_+^{/} dr_+^{/} = \frac{\pi}{2}[r_{in}^2 - r_{out}^2] = -\frac{1}{2}\Delta S_{BH} \quad \text{.............................(6.4.28)}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, Q-q) - S_{BH}(M,Q) = \pi[r_{out}^2 - r_{in}^2]$ is the difference of the Bekenstein-Hawking entropies of the Kerr-Newman blackhole before and after the particle emission. From equation (6.4.18) we obtain the tunneling rate

$\Gamma \sim e^{\Delta S_{BH}}$ ....(6.4.29)

Now , if we expand ΔS_{BH} in terms of ω, q and take only the first order term then

$\Delta S_{BH} = -\beta(\omega - \omega_0)$ where $\beta = \frac{2\pi[M + \sqrt{M^2 - Q^2 - a^2}]^2}{\sqrt{M^2 - Q^2 - a^2}}$ is the inverse Hawking temperature and $\omega_0 = \frac{Qq}{M + \sqrt{M^2 - Q^2 - a^2}}$. Then from (6.4.29) the emission rate becomes

$\Gamma \sim e^{-\beta(\omega - \omega_0)}$ ....(6.4.30).

Also equation (6.4.30) indicates that when the energy conservation , the angular momentum conservation and the electric charge conservation as well as the particle's self-gravitation are taken into account , the tunneling rate is related to the change of blackhole entropy during the process of the particle's emission and the radiant spectrum is not precisely thermal.

6.5 Charged particle tunneling from non-accelerating and rotating blackholes with electric and magnetic charges:

6.5.1 Non-accelerating and rotating blachkholes with electric and magnetic charge: The Plebanski-Demianski [169,170,171] metric covers a large family of spacetimes which include, among others, the well known blackhole solutions like Schwarzschild, Reissner-Nordström , Kerr, Kerr-Newman, Kerr-NUT, Kerr-Newman-NUT and many others. Here we study a special case of this family of blackholes with rotation but non-accelerating with electric and magnetic charges. The metric of such kind of blackhole is given by [172]

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\rho^2} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta[(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta]}{\rho^2} d\phi^2 - \frac{2a\sin^2\theta[(r^2 + a^2) - \Delta]}{\rho^2} dt d\phi$$

...................................(6.5.1)

where $\Delta = r^2 + a^2 + e^2 + g^2 - 2Mr,\quad \rho^2 = r^2 + a^2\cos^2\theta$. Here M is the mass of the blackhole, e and g are the electric and magnetic charges respectively, a is the angular momentum per unit mass. The event horizon equations are given by $\Delta = 0$ which gives

$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2 - g^2}$ ....(6.5.2)

The event horizon area of this blackhole is given by

$A = 4\pi(r_+^2 + a^2)$ ....(6.5.3)

and Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4} = \pi(r_+^2 + a^2) = \pi[2M^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2} - e^2 - g^2]$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.4)$$

Also the electric potential and magnetic potential are given by[180] $A_\mu = (A_t, 0, 0, A_\phi)$ and $B_\mu = (B_t, 0, 0, B_\phi)$ respectively.

Here,

$$A_t = -\frac{er}{\rho^2}, \qquad A_\phi = \frac{e\, r\, a \sin^2\theta}{\rho^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.5)$$

$$B_t = -\frac{gr}{\rho^2}, \qquad B_\phi = \frac{gra\sin^2\theta}{\rho^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.6)$$

6.5.2 Dragging coordinate system and infinite red- shift surface of Non-accelerating and rotating blachkholes with electric and magnetic charge:

The infinite red shift surface is given by $g_{00} = 0$ which gives

$$r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta - e^2 - g^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.7)$$

Obviously the infinite red shift surface does not coincide with the event horizon surface , which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Also there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (6.5.1) into a dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.8)$

where Ω is the angular velocity.

For the metric (6.5.1) we have,

$$g_{00} = \frac{-(\Delta - a^2\sin^2\theta)}{\rho^2}, \quad g_{11} = \frac{\rho^2}{\Delta}, \quad g_{22} = \rho^2, \quad g_{33} = \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]}{\rho^2}$$

$$g_{03} = \frac{-a\sin^2\theta[(r^2+a^2) - \Delta]}{\rho^2}$$

From (6.5.8), $\Omega = \frac{d\phi}{dt} = \frac{a[(r^2+a^2) - \Delta]}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}$ $\ldots\ldots\ldots\ldots\ldots\ldots(6.5.9)$

At the horizon the angular velocity becomes,

$$\Omega_h = \frac{a}{r_+^2 + a^2} \qquad \text{.............................(6.5.10)}$$

The line element (6.5.1) in the dragging coordinate system becomes,

$$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dr^2 + g_{22} d\theta^2 \qquad \text{...................(6.5.11)}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]}$.

The line element (6.5.11) represents a 3-dimensional hypersurface of 4-dimensional spacetime. The infinite red-shift surface now coincide with the event horizon surface in the dragging coordinate system. So the geometrical optical limit can be applied now.In the dragging coordinate system the electric potential and the magnetic potential can be given by[178]

$$A_t' = -\frac{(r^2+a^2)er}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \qquad \text{...................(6.5.12)}$$

$$B_t' = -\frac{(r^2+a^2)gr}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \qquad \text{...(6.5.13)}$$

6.5.3 Painleve-like coordinate transformation and radial geodesics of non-accelerating and rotating blachkholes with electric and magnetic charges :

To investigate the Hawking radiation as tunneling process it is necessary to eliminate coordinate singularity at the event horizon. In the expression (6.5.11) , there still exists coordinate singularity at the event horizon in the dragging coordinate system. So we continue performing a general Painleve coordinate transformation[173]. This transformation can be done by

$$dt \rightarrow dt + F(r,\theta)dr + G(r,\theta)d\theta \qquad \text{..............................(6.5.14)}$$

where $F(r,\theta)$ and $G(r,\theta)$ are two determined functions of r and θ , and satisfy the integrability condition ,

$$\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r} \qquad \text{.........................(6.5.15)}$$

Thus from (6.5.11) we obtain,

$$ds^2 = \hat{g}_{00}\, dt^2 + \left\{ \hat{g}_{00} F^2(r,\theta) + g_{11} \right\} dr^2 + \left\{ \hat{g}_{00} G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2G(r,\theta)\hat{g}_{00}\, dt d\theta$$

$$+ 2F(r,\theta)G(r,\theta)\hat{g}_{00}\, dr d\theta + 2F(r,\theta)\hat{g}_{00}\, dt dr$$

.................................(6.5.16)

We demand that constant time- slices are flat Euclidean space in radial. So we set,

$$\hat{g}_{00} F^2(r,\theta) + g_{11} = 1$$

$$\Rightarrow F(r,\theta) = \pm\sqrt{\frac{1-g_{11}}{\hat{g}_{00}}} \qquad \text{.................................(6.5.17)}$$

From equation (6.5.15), $G(r,\theta) = \int \frac{\delta F(r,\theta)}{\delta\theta} dr + C(\theta)$ ....(6.5.18)

where $C(\theta)$ is an arbitrary analytic function of θ .

Substituting the value of $F(r,\theta)$ into equation (6.5.16) we get,

$$ds^2 = \hat{g}_{00}\, dt^2 + dr^2 + \left\{ \hat{g}_{00}\, G^2(r,\theta) + g_{22} \right\} d\theta^2 + 2\sqrt{\hat{g}_{00}(1-g_{11})}\; G(r,\theta)\, dr\, d\theta$$

$$+ 2\,\hat{g}_{00}\, G(r,\theta)\, dt\, d\theta \pm 2\sqrt{\hat{g}_{00}(1-g_{11})}\; dt\, dr$$

....................(6.5.19)

The positive sign $(+)$ denotes the spacetime line element of the outgoing particle and the minus sign $(-)$ denotes the spacetime line element of the ingoing particles at the horizon.

According to Landau's theory of the coordinate clock synchronization[174] in a spacetime decomposed in 3+1 dimension, the difference of coordinate times of two events taking place simultaneously in different place is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}}\, dx^i \qquad (i = 1,2,3) \qquad \text{..........................(6.5.20)}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy[175]

$$\frac{\delta}{\delta x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\delta}{\delta x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad , \ (i,j = 1,2,3) \qquad \text{........................(6.5.21)}$$

Now the metric (6.5.19) in the new coordinate system , has a number of attractive features : (1) the metric is well-behaved at the event horizon; (2) the time coordinate t represents the local proper time for radially free-falling observers; (3) the hypersurfaces of constant time-slices are just flat Euclidean space in the oblate spheroidal coordinates; (4) by substituting the components of the metric (6.5.19) into equation (6.5.21), we see that the metric satisfy the Landau's condition of the coordinate clock synchronization $\frac{\delta F(r,\theta)}{\delta\theta} = \frac{\delta G(r,\theta)}{\delta r}$; (5) the infinite red-shift surface coincide with the event horizon surface so that the WKB approximation can be used.

These attractive features are very advantageous for us to discuss Hawking radiation as tunneling and to do an explicit computation of the tunneling probability at the event horizon.

Since the charged massive quanta does not follow the radial null geodesics , so we consider the outgoing particle is a massive shell (de Broglie s-wave). According to de Broglie hypothesis , this massive shell is a sort of de Broglie s-wave. The approximation wave equation is given by [176],

$$\psi(r,t) = C e^{i\left(\int_{r_i-\varepsilon}^{r} P_r dr - \omega t\right)} \quad \text{..............................(6.5.22)}$$

where $r_i - \varepsilon$ is the initial location of the particle.

If we let $\int_{r_i-\varepsilon}^{r} P_r dr - \omega t = \phi_0$ ....(6.5.23)

then we have $\frac{dr}{dt} = \dot{r} = \frac{\omega}{k}$ ....(6.5.24)

where k is the de Broglie wave number.

Comparing equation (6.5.24) with the definition of the phase velocity we know that $\dot{r}$ is the phase velocity of the de Broglie wave. Unlike the electromagnetic wave , the phase velocity v_p of the de Broglie wave is not equal to the group velocity v_g. The definition and relationship between them are,

$$v_p = \frac{dr}{dt} = \dot{r} = \frac{\omega}{k} \quad \text{...............................(6.5.25)}$$

$$v_g = \frac{dr_c}{dt} = \frac{d\omega}{dt} \quad \text{...............................(6.5.26)}$$

$$v_p = \frac{1}{2} v_g \quad \text{...............................(6.5.27)}$$

Since the tunneling across the barrier is an instantaneous process, there are two simultaneous events during the process , one is particle tunneling into the barrier and another is particle tunneling out the barrier. According to Landau's theory of the coordinate clock synchronization[174], the difference of coordinate times of these two simultaneous events is

$$dt = -\frac{\hat{g}_{0i}}{\hat{g}_{00}} dx^i = -\frac{\hat{g}_{01}}{\hat{g}_{00}} dr_c \qquad (d\theta = d\phi = 0) \quad \text{..................(6.5.28)}$$

where r_c denote the location of the tunneling particle. The group velocity is

$$v_g = \frac{dr_c}{dt} = -\frac{\hat{g}_{00}}{\hat{g}_{01}} \qquad \text{.................................(6.5.29)}$$

and the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2}v_g = -\frac{1}{2}\frac{\hat{g}_{00}}{\hat{g}_{01}} = -\frac{1}{2}\frac{\hat{g}_{00}}{\sqrt{\hat{g}_{00}(1-g_{11})}} = \frac{\Delta\rho}{2\sqrt{(\rho^2-\Delta)[(r^2+a^2)^2-\Delta a^2\sin^2\theta]}}$$

..............................(6.5.30)

6.5.4 Tunneling rate of charged particles from Non-accelerating and rotating blachkholes with electric and magnetic charges:

We consider the spacetime as dynamical and incorporating the self-gravitation effect of the tunneling particle when the energy conservation, angular momentum conservation , the electric charge conservation and magnetic charge conservation are taken into account. We assume that the total ADM mass ,angular momentum and charge of the hole-particle system are held fixed whereas the mass ,angular momentum and the charge of the hole are allowed to fluctuate, then the mass ,the electric charge and the magnetic charge will become $M-\omega$, $e-e_1$ and $g-g_1$ when a particle with energy ω , electric charge e_1 and magnetic charge g_1 has tunneled from the event horizon. So considering the charged massive particle tunnel's out from the event horizon along the radial direction, we should modify the equation (6.5.30) as ,

$$\dot{r} = \frac{\bar{\Delta}\rho}{2\sqrt{(\rho^2-\bar{\Delta})[(r^2+a^2)^2-\bar{\Delta}a^2\sin^2\theta]}} \qquad \text{.........................(6.5.31)}$$

where $\bar{\Delta} = r^2+a^2+(e-e_1)^2+(g-g_1)^2-2(M-\omega)r$ is the horizon equation after the emission of the particle with energy ω , electric charge e_1 and magnetic charge g_1.

We taken into account the effect of the electro-magnetic field to investigate the tunneling of charged particle. That is , the matter- gravity system consists of the blackhole and the electro-magnetic field outside the hole. We write the Lagrangian function of the matter-gravity system as

$$L = L_m + L_e \qquad \text{......................(6.5.32)}$$

where $L_e = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian function of the electro-magnetic field corresponding to the generalized coordinates described by the equation

(6.5.12) and (6.5.13)in the dragged Painleve-Gullstrand coordinate system [184].We can find that the generalized coordinate is an ignorable coordinate. In addition , the coordinate ϕ does not appear in the line element expressions (6.5.11) and (6.5.19). In order to eliminate these two degrees of freedom completely, the action for the classically forbidden trajectory should be written as $S = \int_{t_{in}}^{t_{out}} (L - P_{A_t} \dot{A}_t - P_{B_t} \dot{B}_t - P_\phi \dot{\phi}) dt$

..............................(6.5.33)

Applying the WKB approximation , the emission rate of the tunneling particle is given by[185]

$\Gamma \sim e^{-2 \operatorname{Im} S}$ ....(6.5.34)

The imaginary part of the action for the charged massive particle is

$$\operatorname{Im} S = \operatorname{Im}\left\{ \int_{r_{in}}^{r_{out}} \left[\int_{(0,0,0)}^{(P_r, P_{A_t}, P_\phi)} (\dot{r}\, dp_r' - \dot{A}_t\, dP_{A_t}' - \dot{B}_t\, dP_{B_t}' - \dot{\phi}\, dP_\phi') \right] \frac{dr}{\dot{r}} \right\}$$

.....................(6.5.35)

where P_r, P_{A_t} $P_{\mathbf{B}_t}$ and P_ϕ are the canonical momentum conjugate to r, A_t B_t and ϕ respectively.

Also,

$r_{in} = M + \sqrt{M^2 - e^2 - g^2 - a^2}$, $r_{out} = (M - \omega) + \sqrt{(M - \omega)^2 - (e - e_1)^2 - (g - g_1)^2 - a^2}$

are the locations of the event horizons before and after the charged particle emission. According to the Hamilton's equations, we have

$$\dot{r} = \frac{dH}{dP_r}\bigg|_{(r;\phi,P_\phi;A_t,P_{A_t};B_t,P_{B_t})} = \frac{d(M-\omega)}{dP_r} \quad \text{..................................(6.5.36)}$$

$$\dot{\phi} = \frac{dH}{dP_\phi}\bigg|_{(\phi;A_t,P_{A_t};B_t,P_{B_t};r,P_r)} = a\bar{\Omega}\frac{d(M-\omega)}{dP_\phi} \quad \text{..............................(6.5.37)}$$

$$\dot{A}_t = \frac{dH}{dP_{A_t}}\bigg|_{(A_t;r,P_r;\phi,P_\phi;B_t,P_{B_t})} = \Phi_e \frac{d(e-e_1)}{dP_{A_t}} \quad \text{.............................(6.5.38)}$$

$$\dot{B}_t = \frac{dH}{dP_{B_t}}\bigg|_{(B_t;r,P_r;\phi,P_\phi;A_t,P_{A_t})} = \Phi_g \frac{d(g-g_1)}{dP_{B_t}} \quad \text{.............................(6.5.39)}$$

where the dragging angular velocity Ω , electric potential Φ_e and magnetic potential Φ_g in the dragging coordinate system are given by

$$\bar{\Omega} = \frac{a(r^2 + a^2 - \bar{\Delta})}{(r^2 + a^2)^2 - \bar{\Delta} a^2 \sin^2\theta} \quad \text{..............................(6.5.40)}$$

$$\Phi_e = -\frac{(e-e_1)r(r^2+a^2)}{(r^2+a^2)^2 - \bar{\Delta} a^2 \sin^2\theta} \quad \text{.............................(6.5.41)}$$

$$\Phi_g = -\frac{(r^2+a^2)(g-g_1)r}{(r^2+a^2)^2 - \bar{\Delta} a^2 \sin^2\theta} \quad \text{................................(6.5.42)}$$

Substituting equation(6.5.31) and (6.5.36)-(6.5.42) into equation (6.5.35), then imaginary part of the action becomes

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}(M-\omega,e-e_1,g-g_1)} \int_{(M,e,g)} [d(M-\omega') - a\Omega' d(M-\omega') - \Phi'_e d(e-e'_1) - \Phi'_g d(g-g'_1)] \frac{dr}{\dot{r}}$$

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}(M-\omega,e-e_1,g-g_1)} \int_{(M,e,g)} [(1-a\Omega')d(M-\omega') - \frac{(r^2+a^2)(e-e'_1)r}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} d(e-e'_1) - \frac{(r^2+a^2)(g-g'_1)r}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} d(g-g'_1)] \frac{dr}{\dot{r}}$$

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}(M-\omega,e-e_1,g-g_1)} \int_{(M,e,g)} [(1-a\Omega')d(M-\omega') - \frac{(r^2+a^2)(e-e'_1)r}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} d(e-e'_1) - \frac{(r^2+a^2)(g-g'_1)r}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} d(g-g'_1)] \frac{2\sqrt{(\rho^2-\Delta')[(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta]}}{\rho\Delta'} dr$$

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}(M-\omega,e-e_1,g-g_1)} \int_{(M,e,g)} [(1-a\Omega')d(M-\omega') - \frac{(r^2+a^2)(e-e'_1)r}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} d(e-e'_1) - \frac{(r^2+a^2)(g-g'_1)r}{(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta} d(g-g'_1)] \frac{2\sqrt{(\rho^2-\Delta')[(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta]}}{\rho(r-r'_+)(r-r'_-)} dr$$

..............................(6.5.43)

where

$$\Delta' = r^2 + a^2 - (M-\omega')r + (e-e'_1)^2 + (g-g'_1)^2 = (r-r'_+)(r-r'_-)$$

$$r'_\pm = (M-\omega') \pm \sqrt{(M-\omega')^2 - (e-e'_1)^2 - (g-g'_1)^2 - a^2}$$

We see that $r = r'_+$ is a simple pole at the event horizon. The integral can be evaluated by deforming the contour around the pole, so as to ensure that the positive energy solution decay in time. Note that all real parts , divergent or not , can be discarded since they only contribute a phase. Doing the r integral first we obtain,

$$\operatorname{Im} S = -\pi \int_{(M,e,g)}^{(M-\omega,e-e_1,g-g_1)} \frac{2(r_+'^2 + a^2)}{(r_+' - r_-')}[(1 - a\Omega'_{r_+'})d(M-\omega')$$

$$- \frac{r_+'(e-e_1')}{(r_+'^2 + a^2)} d(e-e_1') - \frac{(g-g_1')r_+'}{(r_+'^2 + a^2)} d(g-g_1')]$$

$$\operatorname{Im} S = -\pi \int_{(M,e,g)}^{(M-\omega,e-e_1,g-g_1)} \frac{r_+'}{(r_+' - r_-')}[2r_+' d(M-\omega')$$

$$- 2(e-e_1')d(e-e_1') - 2(g-g_1')d(g-g_1')]$$

.....................................(6.5.44)

Now from $r_+' = (M-\omega') + \sqrt{(M-\omega')^2 - (e-e_1')^2 - (g-g_1')^2 - a^2}$ we obtain the identity

$$(r_+' - r_-')dr_+' = 2r_+' d(M-\omega') - 2(e-e_1')d(e-e_1') - 2(g-g_1')d(g-g_1')$$

.................................(6.5.45)

Using the identity(6.5.45) into equation (6.5.44) we can easily finish the integration and yields a simple expression

$$\operatorname{Im} S = -\pi \int_{r_{in}}^{r_{out}} r_+' dr_+' = \frac{\pi}{2}[r_{in}^2 - r_{out}^2] = -\frac{1}{2}\Delta S_{BH} \quad \text{..............................(6.5.46)}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, e-e_1, g-g_1) - S_{BH}(M,e,g) = \pi[r_{out}^2 - r_{in}^2]$ is the difference of the Bekenstein-Hawking entropies of the blackhole before and after the particle emission. From equation (6.5.34) we obtain the tunneling rate

$$\Gamma \sim e^{\Delta S_{BH}} \quad \text{.............................(6.5.47)}$$

Equation (6.5.47) indicates that the tunneling rate is related to the difference of the Bekenstein-Hawking entropies of the blackhole before and after the emission of the shell of energy ω, electric charge e_1 and magnetic charge g_1.

CHAPTER SEVEN
HAWKING RADIATION VIA HAMILTON-JACOBI METHOD

7.1 Hamilton-Jacobi Method : In the past two decades, a lot of researchers have investigated Hawking radiation of blackholes [4,9]. Most of them rely on the quantum field theory on the fixed background spacetime and derived radiation spectrum is pure thermal[186]. Parikh and Wilczek [14] employed the semi-classical tunneling method to research the Hawking radiation of Schwarzschild and Reissner-Nordström blackhole. Their research has shown that the derived radiation spectrum is not pure thermal and the tunneling probability is related to the change of Bekenstein-Hawking entropy when the self- gravitation interaction and energy conservation are taken into account. In their methodology, the key point is the find the motion of equation of the emitted particle and to calculate the action by Hamilton equation .Thus one has to perform Painleve-Gullstrand coordinate transformation. Following the method, great effort has been devoted to the Hawking radiation of massless particle and massive charged particle particles, which has effective significance for the furthermore cognition and research on blackhole.

Recently, M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini[187] developed a method to study the Hawking radiation of blackholes , which is known as Hamilton-Jacobi method and focuses on the calculation of the particle action via the Hamilton-Jacobi equation to investigate Hawking radiation of blackholes. The main characteristics of this method is the covariant treatment of the horizon singularity by using the spatial proper distance . However, the derived radiation spectrum is pure thermal since they have lost the sight of the self-gravitation of the particle[188]. In fact the background spacetime of blackholes is not fixed and the self-gravitation interaction should be taken into account during the research of Hawking radiation[189]. Now we investigate the Hawking radiation via Hamilton-Jacobi method of some different kinds of blackholes.

7.2 Hawking radiation as tunneling via Hamilton- Jacobi method from Schwarzschild Blackhole: The line element of Schwarzschild blackhole is given by

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(with $G = c = 1$)

$$ds^2 = -f(r)dt^2 + f^{-1}(r)\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{.................................(7.2.1)}$$

where $f(r) = 1 - \frac{2M}{r}$ and $r_h = 2M = radius\ of\ the\ event\ horizon.$

We consider a scalar particle moving in this spacetime without its self-gravitation.

7.2.1 Tunneling rate of massless particles from Schwarzschild Blackhole: Within the semi-classical approximation , the classical I of the particle satisfies the relativistic Hamilton-Jacobi equation [197]

$$g^{\mu\nu}\delta_\mu I\,\delta_\nu I + m^2 = 0 \quad \text{...(7.2.2)}$$

where m is the mass of the scalar particle and $g^{\mu\nu}$ are the inverse metric tensor components obtained from (7.2.1) namely

$g^{00} = -\frac{1}{1-\frac{2M}{r}}$, $g^{11} = 1 - \frac{2M}{r}$, $g^{22} = \frac{1}{r^2}$, $g^{33} = \frac{1}{r^2\sin^2\theta}$ and other components are zero.

Consider equation (7.2.1) , the equation (7.2.2) can be written as

$$-\frac{1}{f(r)}(\frac{\delta I}{\delta t})^2 + f(r)(\frac{\delta I}{\delta r})^2 + \frac{1}{r^2}(\frac{\delta I}{\delta\theta})^2 + \frac{1}{r^2\sin^2\theta}(\frac{\delta I}{\delta\phi})^2 + m^2 = 0 \quad \text{...........................(7.2.3)}$$

By considering the axial symmetry of the blackhole spacetime , we carry out the separation of variable to (7.2.3) as

$I = -\omega t + W(r) + J(x^i)$

Therefore we have,

$$\delta_t I = -\omega, \qquad \delta_r I = W'(r), \quad \delta_\theta I = J_\theta\ , \ \delta_\phi I = J_\phi \quad \text{.................................(7.2.4)}$$

where J_θ and J_ϕ are constant respectively.

From (7.2.3) we have

$$-\frac{1}{f(r)}\omega^2 + f(r)W'^2(r) + \frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\phi^2 + m^2 = 0 \quad \text{...........................(7.2.5)}$$

From above equation we obtain,

$$W = \int \frac{\sqrt{\omega^2 - f(r)\left\{\frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\phi^2 + m^2\right\}}}{f(r)}dr$$

So we have ,

$$I = -\omega t + \int \frac{\sqrt{\omega^2 - f(r)\left\{\frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\phi^2 + m^2\right\}}}{f(r)}dr + J(x') \quad \text{..............................(7.2.6)}$$

By directly use Feynman prescription to deal with the integral over the coordinate , we will get one half of the correct one, that is $\operatorname{Im} I = \operatorname{Im} W = \pi r_h \omega$.[190]

However , if the above calculation making use of the isotropic coordinate defined by

$$t \to t, \quad r \to \rho \text{ , } \ln \rho = \int \frac{dr}{r\sqrt{f(r)}} \quad \text{................................(7.2.7)}$$

The metric assumes the form[190]

$$ds^2 = -f(r(\rho))\,dt^2 + \frac{r^2(\rho)}{\rho^2}\{d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ \}$$

$$= -dt^2 \frac{(1-\frac{r_h}{4\rho})^2}{(1+\frac{r_h}{4\rho})^2} + (1+\frac{r_h}{4\rho})^4(d\rho^2 + \rho^2 ds_2^2) \quad \text{...................................(7.2.8)}$$

In this system of coordinate, the spatial metric is no longer singular at the horizon. This form of metric is still static , but with a radial part regular at the horizon $\rho = r_h$. We may apply again similar formula as (7.2.6) deforming the contour and a direct computation gives the correct result $\operatorname{Im} I = \operatorname{Im} W = 2\pi r_h \omega$.

The reason of this discrepancy can be understood observing that in a curved manifold , the non-locally integrable function $\frac{1}{r}$ does not lead to a covariant distribution $\frac{1}{(r+i.0)}$. Because the result above is not invariant under changes of coordinate within a time slice , we introduce the proper spatial distance defined by the spatial metric

$$d\sigma^2 = f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \text{....................................(7.2.9)}$$

so the radial part of the action can read as

$$W(\sigma) = \int \frac{d\sigma\sqrt{\omega^2 - f(r(\sigma))(\frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\phi^2 + m^2)}}{\sqrt{f(r(\sigma))}} \quad \text{.......................(7.2.10)}$$

Using near the horizon approximation

$f(r) = f'(r_h)(r - r_h) + \text{.........................}$

We get the invariant result

$$I = \frac{2\pi i\omega}{f'(r_h)} + (real\ \ contribution) = 4\pi iM\omega + (real\ \ contribution) \quad \text{.....................(7.2.11)}$$

And the semi-classical emission rate

$\Gamma \sim e^{-2\operatorname{Im} I} = e^{-8\pi M\omega}$ ....(7.2.12)

From the above equation we can easily obtain the Boltzman factor $\beta = 8\pi M$.

Now if we take into the self-interaction of the particles; when a particle of energy ω emits throughout the event horizon then due to energy conservation, the mass of the blackhole will be $M-\omega$ and event horizon will change from $r = 2M$ to $r = 2(M-\omega)$. From (7.2.12) we have

$$\Gamma \sim e^{-2\operatorname{Im} I} = e^{-8\pi M\omega} = e^{-4\pi(2M-\omega)\omega} = e^{4\pi\omega^2 - 8\pi M\omega} = e^{\Delta S_{BH}} \quad \text{.....................(7.2.13)}$$

where $\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M) = 4\pi\omega^2 - 8\pi M\omega$ is the difference of Bekenstein-Hawking entropies before and after the emission of particles. This result is accordance with Parikh-Wilczek's result.

7.3 Hawking radiation as tunneling via Hamilton-Jacobi method from Reissner-Nordstr$\ddot{o}$ m blackhole: The line element of Reissner-Nordstr$\ddot{o}$ m blackhole is given by,

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Or, $ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ ....(7.3.1)

With the electromagnetic potential $A_\mu = (A_t, 0, 0, 0)$

Where $\Delta = 1 - \dfrac{2M}{r} + \dfrac{Q^2}{r^2}$, $A_t = -\dfrac{Q}{r}$ ....(7.3.2)

and $r_h = M + \sqrt{M^2 - Q^2}$ is the location of the event horizon. In Parikh=Wilczek's method one should adopt the Painleve-like coordinate transformation in order to the line element is well behaved at the event horizon. And the motion of the equation of the particle should be also calculated in order to calculate the action . But these can be avoided in Hamilton-Jacobi method. Near the event horizon of the blackhole we can get the line element as [191]

$$ds^2 = -\Delta_{,r}(r_h)(r-r_h)dt^2 + [\Delta_{,r}(r_h)(r-r_h)]^{-1}dr^2 + r_h^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{.................(7.3.3)}$$

where $\Delta_{,r}(r_h) = \frac{\delta\Delta}{\delta r}\Big|_{r=r_h}$.

7.3.1 Tunneling rate of charged particles via Hamilton-Jacobi method from Reissner-Nordström blackhole The classical action I of the charged particle satisfies the relativistic Hamilton-Jacobi equation [197]

$$g^{\mu\nu}(\delta_\mu I - qA_\mu)(\delta_\nu I - qA_\nu) + u^2 = 0 \quad \text{.......................................(7.3.4)}$$

where u and q are the mass and charge of the particle and is the inverse tensor obtained using by the line element (7.3.3) and substituting it into equation (7.3.4) we get [191]

$$-\frac{1}{\Delta_{,r}(r_h)(r-r_h)}(\delta_t I - qA_t)^2 + \Delta_{,r}(r_h)(r-r_h)(\delta_r I)^2 + \frac{1}{r_h^2}[(\delta_0 I)^2 + \frac{1}{\sin^2\theta}(\delta_\phi I)^2] + u^2 = 0$$

$$\text{...(7.3.5)}$$

Now considering the axial symmetry of the blackhole , we carry on the following separation variable

$$I = -\omega t + W(r) + Y(\theta,\phi) \quad \text{...(7.3.6)}$$

where ω is the energy of the emitted particles, $W(r)$ is the generalized momentum in radial.

From equation (7.3.5) and (7.3.6) we can obtain

$$W(r) = \frac{1}{\Delta_{,r}(r_h)}\int \frac{dr}{r-r_h}\sqrt{(\omega + qA_t)^2 - \frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}[(\delta_0 Y - \frac{\delta_\phi Y}{\sin^2\theta}) + u^2]}$$

$$\text{...................................(7.3.7)}$$

Introducing the proper spatial distance [187,192] which is defined by

$$d\sigma^2 = [\Delta_{,r}(r-r_h)]^{-1}dr^2 + r_h^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{.............................(7.3.8)}$$

Limiting to the s-wave contribution that is the bulk of the particle emission[191] we get

$$\sigma = \frac{2}{\Delta_{,r}(r_h)}\sqrt{r-r_h} \quad \text{................................(7.3.9)}$$

Then equation (7.3.7) can be rewritten as

$$W(\sigma)=\frac{2}{\Delta_{,r}(r_h)}\int\frac{d\sigma}{\sigma}\sqrt{(\omega+qA_t)^2-\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}[(\delta_\theta Y+\frac{\delta_\phi Y}{\sin^2\theta})+u^2]}$$

....(7.3.10)

The solution is singular at $\sigma=0$ which corresponds to the event horizon. Thus deforming the integration contour from the real σ – axis to the lower complex σ – plane that avoid the pole $\sigma=0$ counter clock wise and using the Feynman prescription at the event horizon , we obtain the imaginary part of the action I as

$$\operatorname{Im} I=\frac{2\pi}{\Delta_{,r}(r_h)}(\omega-q\frac{Q}{r_h}) \qquad (7.3.11)$$

Now using WKB approximation , we can get the tunneling probability of the emitted particle and find the radiation spectrum being pure thermal. However, the recent result shows that the radiation spectrum deviates from pure thermal one and the tunneling probability are related to the change of Bekenstein-Hawking entropy before and after the particle emitted. The reason of pure thermal spectrum is that the self-gravitation interaction of the emitted particle was not considered in this process. Now if we taking the self-gravitation interaction as well as the conservation of energy and charge into account, we turn to return to the Hawking radiation of Reissner-Nordström blackhole. Let we fix the mass and charge of the total spacetime and allow those of the blackhole to be varied, when a particle of energy ω and charge q tunnels out , the parameters of the mass and charge in equation(7.3.11) should be changed and the modified imaginary part of the action is

$$\begin{aligned}\operatorname{Im} I&=\int_{(0,0)}^{(\omega,q)}\frac{2\pi}{\Delta'_{,r}(r'_h)}(d\omega'-\frac{Q-q'}{r'_h}dq')\\&=-\int_{(M,Q)}^{(M-\omega,Q-q)}\frac{2\pi}{\Delta'_{,r}(r_h)}[d(M-\omega')-\frac{Q-q'}{r'_h}d(Q-q')]\end{aligned}$$

.............................(7.3.12)

where ,
$$\Delta'_{,r}(r'_h)=\frac{2(M-\omega')}{r'_h}-\frac{2(Q-q')^2}{r'^3_h} \qquad (7.3.13)$$
$$r'_h=M-\omega'+\sqrt{(M-\omega')^2-(Q-q')^2}$$

Substituting equation (7.3.13) into equation (7.3.12) we get

$$\operatorname{Im} I = -\int_{(M,Q)}^{(M-\omega,Q-q)} \frac{\pi r_h'^3}{(M-\omega')r_h' - (Q-q')^2}\left[d(M-\omega') - \frac{Q-q'}{r_h'}d(Q-q')\right]$$

Finishing the integral we obtain

Finishing the integral we have,

$$\operatorname{Im} I = -\frac{\pi}{2}\left[\left\{(M-\omega) + \sqrt{(M-\omega)^2 - (Q-q)^2}\right\}^2 - \left\{M + \sqrt{M^2 - Q^2}\right\}^2\right] = -\frac{1}{2}\Delta S_{BH}$$

……………………………..(7.3.14)

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission.

The tunneling rate is therefore

$$\Gamma \sim e^{-2\operatorname{Im} I} = e^{\Delta S_{BH}} \quad \text{……………………………….(7.3.15)}$$

The result show that the tunneling probability is related to the change of Bekenstein-Hawking entropy and the radiation spectrum deviates from pure thermal one , which supports the Parikh and Wilczek's result.

7.4 Hawking radiation as tunneling via Hamilton-Jacobi method from Kerr blackhole: : The line element of Kerr metric in the Boyer-Lindquist coordinate system is given by,

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - a\,dt]^2$$

…………………………(7.4.1)

where

$$\Delta = r^2 - 2Mr + a^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta \quad \text{………………………………………(7.4.2)}$$

$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum and r is the radial distance from the center of the body. The equation of the event horizon is given by,

$\Delta = 0$ which gives, $r_{\pm} = M \pm \sqrt{M^2 - a^2}$, $M^2 > a^2$.

Since , the event horizon $r_{\pm} = M \pm \sqrt{M^2 - a^2}$ does not coincide with the infinite red-shift surface $r_{\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta}$, which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. So we adopt dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ ……………………(7.4.3)

where Ω is the angular velocity.

For the metric (7.4.1) we have,

$$g_{00} = \frac{-(\Delta - a^2 \sin^2\theta)}{\rho^2},\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \frac{\sin^2\theta[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta]}{\rho^2}$$

$$g_{03} = \frac{-a\sin^2\theta[(r^2+a^2)-\Delta]}{\rho^2}$$

From (7.4.3), $$\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}} = \frac{a[(r^2+a^2)-\Delta]}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$$

..............................(7.4.4)

At the horizon the angular velocity becomes,

$$\Omega_h = \frac{a}{r_+^2 + a^2} \qquad \text{..........................(7.4.5)}$$

The line element (7.4.1) in the dragging coordinate system becomes

$$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dr^2 + g_{22} d\theta^2 \qquad \text{..................(74.6)}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$.

The area and Bekenstein-Hawking entropy corresponding to the outer event horizon of the blackhole is given by

$$A = \int \sqrt{-g}\ d\theta d\phi = 4\pi(r_+^2 + a^2)$$

$$S_{BH} = \frac{A}{4} = \pi(r_+^2 + a^2) \qquad \text{..(7.4.7)}$$

7.4.1 Tunneling rate of massless particle from Kerr blackhole via Hamilton-Jacobi Method:

The classical action I of the radiation particle satisfies the relativistic Hamilton-Jacobi equation as [197]

$$g^{\mu\nu}\delta_\mu I \delta_\nu I + u^2 = 0 \qquad \text{..(7.4.8)}$$

where u is the mass of the emitted particle and $g^{\mu\nu}$ are the inverse metric tensor obtained from (7.4.1)as

$$\hat{g}^{00} = -\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Delta\rho^2},\ g^{11} = \frac{\Delta}{\rho^2},\ g^{22} = \frac{1}{\rho^2}$$

....(7.4.9)

and other components are zero.

Putting these value into the equation (7.4.8) we have

$$\hat{g}^{00}(\frac{\delta I}{\delta t})^2 + g^{11}(\frac{\delta I}{\delta r})^2 + g^{22}(\frac{\delta I}{\delta \theta})^2 + u^2 = 0$$

$$or,\ -\frac{1}{P(r,\theta)}(\frac{\delta I}{\delta t})^2 + M(r,\theta)\,(\frac{\delta I}{\delta r})^2 + C(r,\theta)\,(\frac{\delta I}{\delta \theta})^2 + u^2 = 0$$

....(7.4.10)

where,

$$P(r,\theta) = \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Delta \rho^2}, \qquad M(r,\theta) = \frac{\Delta}{\rho^2},\ C(r,\theta) = \frac{1}{\rho^2}$$

..................................(7.4.11)

Now considering the axial symmetry of the blackhole , we carry on the following separation variable

$I = -\omega t + W(r,\theta) + j\phi$ ....(7.4.12)

where ω is the energy of the emitted particle ,$W(r,\theta)$ is the generalized momentum and j is the angular momentum with respect to the ϕ-axis. From (7.4.12)

$$\frac{\delta I}{\delta t} = -\omega + j\frac{\delta \phi}{\delta t} = -\omega + j\Omega$$

...................................(7.4.13)

$$\frac{\delta I}{\delta r} = \frac{\delta W}{\delta r},\ \frac{\delta I}{\delta \theta} = \frac{\delta W}{\delta \theta}$$

Substituting these into equation(7.4.10) we can obtain

$$\frac{\delta W}{\delta r} = \frac{1}{\sqrt{M(r,\theta)\,P(r,\theta)}}\sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W}{\delta \theta})^2 + u^2\right\}}$$

......................................(7.4.14)

From above equation we can learn that the imaginary part of the emitted particle 's action is only produced from the pole at the event horizon [193]. According to the reference [185] for getting the correct result , the proper spatial distance should be introduced, which is defined by

$d\sigma^2 = \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2$ ....(7.4.15)

Since there is no motion in the θ-direction ,so we have

$\sigma = \int \frac{1}{\sqrt{M(r,\theta)}}dr$ ....(7.4.16)

since $M(r,\theta) = \frac{\Delta}{\rho^2}$

Now at the horizon

$$\left.\begin{aligned} P(r,\theta) &= P'(r_+,\theta)(r - r_+) + \ldots \\ M(r,\theta) &= M'(r_+,\theta)(r - r_+) + \ldots \end{aligned}\right\}$$ ....(7.4.17)

Where $\dfrac{\delta P(r,\theta)}{\delta r}\Big|_{r=r_+} = P'(r_+,\theta)$

$\dfrac{\delta M(r,\theta)}{\delta r}\Big|_{r=r_+} = M'(r_+,\theta)$

Hence $\sigma = \int \dfrac{1}{\sqrt{M'(r_+,\theta)(r-r_+)+\dots\dots}}dr$

$$= \frac{2}{\sqrt{M'(r_+,\theta)}}\sqrt{r-r_+} \quad \dots\dots(7.4.18)$$

Again $d\sigma = \dfrac{dr}{\sqrt{M(r,\theta)}} = \dfrac{1}{\sqrt{M'(r_+,\theta)(r-r_+)+\dots\dots}}dr$

$$\dots\dots(7.4.19)$$

Or, $\dfrac{d\sigma}{\sigma} = \dfrac{dr}{2(r-r_+)}$ ....(7.4.20)

From equation (7.4.14) we have

$$\delta W = \frac{2(r-r_+)}{\sqrt{\{M'(r_+,\theta)(r-r_+)+\dots\dots\}\{P'(r_+,\theta)(r-r_+)+\dots\dots\}}}$$

$$\frac{d\sigma}{\sigma}\sqrt{(\omega - j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W(r,\theta)}{\delta\theta}\right)^2 + u^2\right\}}$$

Or, $W(\sigma) = \dfrac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\displaystyle\int\frac{d\sigma}{\sigma}\sqrt{(\omega - j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W(r,\theta)}{\delta\theta}\right)^2 + u^2\right\}}$

$$\dots\dots(7.4.21)$$

where $\Omega_+ = \dfrac{a}{r_+^2 + a^2}$ is the velocity at the event horizon and the solution is singular at $\sigma = 0$ which corresponds to the event horizon.

Also $\dfrac{d\sigma}{\sigma} = \dfrac{dr}{2(r-r_+)}$. Therefore from equation (7.4.21) we can obtain

$$W = \int \frac{2dr}{2(r-r_+)\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\sqrt{(\omega - j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W}{\delta\theta}\right)^2 + u^2\right\}}$$

$$= \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{dr}{(r-r_+)}\sqrt{(\omega - j\Omega_+)^2 - \{P'(r_+,\theta)(r-r_+)+\dots\dots\}(0+u^2)}$$

since $\dfrac{\delta W}{\delta\theta} = 0$.

Or, $W = \dfrac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\displaystyle\int\frac{(\omega - j\Omega_+)}{(r-r_+)}\sqrt{1-b(r-r_+)}\,dr$ where $b = \dfrac{P'(r_+,\theta)u^2}{(\omega - j\Omega_+)^2}$

$$= \frac{(\omega - j\Omega_+)}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{\sqrt{1-b(r-r_+)}\;d(r-r_+)}{(r-r_+)}$$

Or, $W = \frac{(\omega - j\Omega_+)}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} \int \frac{\sqrt{1-bz}\ dz}{z}$ where $z = r - r_+$

Here the singularity is at $z = 0$. Applying the Cauchy integral formula we can obtain

$$\operatorname{Im} W = \operatorname{Im} I = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(\omega - j\Omega_+) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(\omega - \frac{ja}{r_+^2 + a^2})$$

....(7.4.22)

Now the temperature over the surface of the blackhole is given by[193]

$$T = \frac{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}{4\pi} = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + a^2}$$

............................(7.4.23)

$$or, \quad \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} = \frac{r_+^2 + a^2}{2(r_+ - M)}$$

From (7.4.22) we have

$$d(\operatorname{Im} I) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(d\omega - \frac{adj}{r_+^2 + a^2})$$

....(7.4.24)

$$or, \quad \operatorname{Im} I = \int \frac{\pi(r_+^2 + a^2)}{r_+ - M}(d\omega - \frac{adj}{r_+^2 + a^2})$$

If we fix the total ADM mass and angular momentum of the spacetime and allow those of the blackhole to vary, then when a particle with energy ω and angular momentum j tunnels out, the mass and angular momentum should be modified. Replacing M by $M - \omega$ and J by $J - j$ we obtain the imaginary part of the actual action as

$$\operatorname{Im} I = \pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + a^2}{r_+' - (M - \omega')}(d\omega' - \frac{adj'}{r_+'^2 + a^2})$$

$$= -\pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + a^2}{r_+' - (M - \omega')}\left\{d(M - \omega') - \frac{ad(J - j')}{r_+'^2 + a^2}\right\}$$

....(7.4.25)

where

$$J - j' = a(M - \omega'), \qquad r_+ = M + \sqrt{M^2 - a^2}$$

$$r_+' = (M - \omega') + \sqrt{(M - \omega')^2 - a^2}$$

Now , $\int_J^{J-j} d(J - j') = [J - j']_J^{J-j} = (J - j) - J \approx J - J = 0$

j is so small that we assume that $J - j \sim J$. Therefore

$$\operatorname{Im} I = -\pi \int_{M}^{M-\omega} \frac{r_+'^2 + a^2}{r_+' - (M-\omega')} \, d(M-\omega')$$

Finishing the integral we have

$$\operatorname{Im} I = -\pi[(M-\omega)\sqrt{(M-\omega)^2 - a^2} + (M-\omega)^2 - M\sqrt{M^2 - a^2} - M^2]$$

The tunneling rate

$$\Gamma \sim e^{-2\operatorname{Im} I} = e^{2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2} - M^2 - M\sqrt{M^2 - a^2}]} = e^{\Delta S_{BH}}$$

where ΔS_{BH} is the difference of Bekenstein-Hawking entropy of the blackhole before and after the particle emission from the blackhole.

7.5 Hawking radiation as Tunneling via Hamilton-Jacobi method from Kerr-NUT blackhole: The line element of Kerr-NUT metric is given by [194]

$$ds^2 = -\frac{\Delta^2}{\rho^2}(dt - P\,d\phi)^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(F + l^2)d\phi - a\,dt]^2$$

....(7.5.1)

where $F = r^2 + a^2$

$$\Delta^2 = r^2 - 2Mr + a^2 - l^2$$

$$\rho^2 = r^2 + (l + a\cos\theta\,)^2 \qquad \text{......................................(7.5.2)}$$

$$a = \frac{J}{M}, \qquad P = a^2\sin^2\theta - 2l\cos\theta$$

Here M is the mass of the body, J is the angular momentum , l is the NUT parameter. The area and Bekenstein-Hawking entropy corresponding to the outer event horizon of the blackhole is given by

$$A = \int\sqrt{-g}\;d\theta d\phi = 4\pi(r_+^2 + a^2 + l^2)$$

$$S_{BH} = \frac{A}{4} = \pi(r_+^2 + a^2 + l^2) \text{ ....(7.5.3)}$$

The infinite red shift surface and the event horizon of the blackhole is given by $r = M + \sqrt{M^2 - a^2\cos^2\theta + l^2}$ and $r = M + \sqrt{M^2 - a^2 + l^2}$ respectively. Obviously they are not coincide to each other which is inconvenient to study the Hawking radiation. So we adopt dragging coordinate system. Thus we perform the dragging coordinate transformation

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}} \text{ ....(7.5.4)}$$

where Ω is the angular velocity, on the line element (7.5.1) and we get

$$ds^2 = -\frac{\Delta^2 \rho^2 \sin^2\theta}{(r^2+a^2+l^2)\sin^2\theta - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2} dt^2 + \frac{\rho^2}{\Delta^2} dr^2 + \rho^2 d\theta^2$$

....(7.5.5)

Now the event horizon and infinite red shift surface are coincident with each other, which means the geometrical optical limit can be applied now. Using WKB approximation we can get the relationship between the tunneling rate and the imaginary part of the radiation particle as $\Gamma \sim e^{-2\,\mathrm{Im}\, I}$.

7.5.1Tunneling rate of massless particles via Hamilton-Jacobi Method from Kerr-NUT blackhole :

The classical action I of the radiation particle satisfies the relativistic Hamilton-Jacobi equation as [197]

$$g^{\mu\nu}\delta_\mu I \delta_\nu I + u^2 = 0 \qquad \text{...(7.5.6)}$$

where u is the mass of the emitted particle and $g^{\mu\nu}$ are the inverse metric tensor obtained from (7.5.1) as

$$\hat{g}^{00} = -\frac{(r^2+a^2+l^2)\sin^2\theta - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}{\Delta^2\rho^2\sin^2\theta}, \quad g^{11} = \frac{\Delta^2}{\rho^2}, \quad g^{22} = \frac{1}{\rho^2}$$

....(7.5.7)

and other components are zero.

Putting these value into the equation (7.5.6) we have

$$\hat{g}^{00}\left(\frac{\delta I}{\delta t}\right)^2 + g^{11}\left(\frac{\delta I}{\delta r}\right)^2 + g^{22}\left(\frac{\delta I}{\delta \theta}\right)^2 + u^2 = 0$$

$$or,\ -\frac{1}{P(r,\theta)}\left(\frac{\delta I}{\delta t}\right)^2 + M(r,\theta)\left(\frac{\delta I}{\delta r}\right)^2 + C(r,\theta)\left(\frac{\delta I}{\delta \theta}\right)^2 + u^2 = 0$$

....(7.5.8)

where,

$$P(r,\theta) = \frac{(r^2+a^2+l^2)\sin^2\theta - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}{\Delta^2\rho^2\sin^2\theta}, \qquad M(r,\theta) = \frac{\Delta^2}{\rho^2}, \quad C(r,\theta) = \frac{1}{\rho^2}$$

.................................(7.5.9)

Now considering the axial symmetry of the blackhole , we carry on the following separation variable

$$I = -\omega t + W(r,\theta) + j\phi \qquad \text{..(7.5.10)}$$

where ω is the energy of the emitted particle ,$W(r,\theta)$ is the generalized momentum and j is the angular momentum with respect to the ϕ-axis. From (7.5.10)

$$\frac{\delta I}{\delta t} = -\omega + j\frac{\delta\phi}{\delta t} = -\omega + j\Omega$$

$$\frac{\delta I}{\delta r} = \frac{\delta W}{\delta r}, \frac{\delta I}{\delta\theta} = \frac{\delta W}{\delta\theta} \qquad \text{..................................(7.5.11)}$$

Substituting these into equation(7.5.8) we can obtain

$$\frac{\delta W}{\delta r} = \frac{1}{\sqrt{M(r,\theta)P(r,\theta)}}\sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W}{\delta\theta})^2 + u^2\right\}}$$

$$\text{.....................................(7.5.12)}$$

From above equation we can learn that the imaginary part of the emitted particle 's action is only produced from the pole at the event horizon [193]. According to the reference [185] for getting the correct result , the proper spatial distance should be introduced, which is defined by

$$d\sigma^2 = \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 \qquad \text{..(7.5.13)}$$

Since there is no motion in the θ – direction ,so we have

$$\sigma = \int \frac{1}{\sqrt{M(r,\theta)}}dr \qquad \text{............................(7.5.14)}$$

since $M(r,\theta) = \dfrac{\Delta^2}{\rho^2}$

Now at the horizon

$$\left.\begin{aligned} P(r,\theta) &= P'(r_+,\theta)(r - r_+) + \ldots\ldots \\ M(r,\theta) &= M'(r_+,\theta)(r - r_+) + \ldots\ldots \end{aligned}\right\} \qquad \text{.............................(7.5.15)}$$

Where $\dfrac{\delta P(r,\theta)}{\delta r}\Big|_{r=r_+} = P'(r_+,\theta)$

$\dfrac{\delta M(r,\theta)}{\delta r}\Big|_{r=r_+} = M'(r_+,\theta)$

Hence $$\sigma = \int \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+) + \ldots\ldots}}dr$$

$$= \frac{2}{\sqrt{M'(r_+,\theta)}}\sqrt{r - r_+} \qquad \text{..(7.5.16)}$$

Again $$d\sigma = \frac{dr}{\sqrt{M(r,\theta)}} = \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+) + \ldots\ldots}}dr$$

$$\text{......................................(7.5.17)}$$

Or, $$\frac{d\sigma}{\sigma} = \frac{dr}{2(r - r_+)} \qquad \text{...(7.5.18)}$$

From equation (7.5.12) we have

$$\delta W = \frac{2(r-r_+)}{\sqrt{\{M'(r_+,\theta)(r-r_+)+..........\}\{P'(r_+,\theta)(r-r_+)+..............\}}}$$

$$\frac{d\sigma}{\sigma}\sqrt{(\omega-j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W(r,\theta)}{\delta\theta})^2+u^2\right\}}$$

Or, $$W(\sigma) = \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{d\sigma}{\sigma}\sqrt{(\omega-j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W(r,\theta)}{\delta\theta})^2+u^2\right\}}$$

....(7.5.19)

where $\Omega_+ = \frac{a}{r_+^2+a^2+l^2}$ is the velocity at the event horizon and the solution is singular at $\sigma = 0$ which corresponds to the event horizon.

Also $\frac{d\sigma}{\sigma} = \frac{dr}{2(r-r_+)}$. Therefore from equation (7.5.19) we can obtain

$$W = \int \frac{2dr}{2(r-r_+)\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\sqrt{(\omega-j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W}{\delta\theta})^2+u^2\right\}}$$

$$= \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{dr}{(r-r_+)}\sqrt{(\omega-j\Omega_+)^2 - \{P'(r_+,\theta)(r-r_+)+........\}(0+u^2)}$$

since $\frac{\delta W}{\delta\theta} = 0$.

Or, $$W = \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{(\omega-j\Omega_+)}{(r-r_+)}\sqrt{1-b(r-r_+)}\,dr$$ where $b = \frac{P'(r_+,\theta)u^2}{(\omega-j\Omega_+)^2}$

$$= \frac{(\omega-j\Omega_+)}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{\sqrt{1-b(r-r_+)}\;d(r-r_+)}{(r-r_+)}$$

Or, $$W = \frac{(\omega-j\Omega_+)}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{\sqrt{1-bz}\;dz}{z}$$ where $z = r - r_+$

Here the singularity is at $z = 0$. Applying the Cauchy integral formula we can obtain

$$\operatorname{Im} W = \operatorname{Im} I = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(\omega-j\Omega_+) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(\omega - \frac{ja}{r_+^2+a^2+l^2})$$

....(7.5.20)

Now the temperature over the surface of the blackhole is given by[193]

$$T = \frac{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}{4\pi} = \frac{1}{2\pi}\frac{r_+-M}{r_+^2+a^2+l^2}$$

.............................(7.5.21)

$$or,\ \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} = \frac{r_+^2+a^2+l^2}{2(r_+-M)}$$

From (7.5.20) we have

$$d(\operatorname{Im} I) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(d\omega - \frac{adj}{r_+^2 + a^2 + l^2})$$

$$or,\ \operatorname{Im} I = \int \frac{\pi(r_+^2 + a^2 + l^2)}{r_+ - M}(d\omega - \frac{adj}{r_+^2 + a^2 + l^2})$$

....(7.5.22)

If we fix the total ADM mass and angular momentum of the spacetime and allow those of the blackhole to vary, then when a particle with energy ω and angular momentum j tunnels out, the mass and angular momentum should be modified. Replacing M by $M-\omega$ and J by $J-j$ we obtain the imaginary part of the actual action as

$$\operatorname{Im} I = \pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + a^2 + l^2}{r_+' - (M-\omega')}(d\omega' - \frac{adj'}{r_+'^2 + a^2 + l^2})$$

$$= -\pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + a^2 + l^2}{r_+' - (M-\omega')}\left\{d(M-\omega') - \frac{ad(J-j')}{r_+'^2 + a^2 + l^2}\right\}$$

....(7.5.23)

where

$$J - j' = a(M-\omega'), \qquad r_+ = M + \sqrt{M^2 - a^2 + l^2}$$

$$r_+' = (M-\omega') + \sqrt{(M-\omega')^2 - a^2 + l^2}$$

Now , $\int_J^{J-j} d(J-j') = [J-j']_J^{J-j} = (J-j) - J \approx J - J = 0$

j is so small that we assume that $J - j \sim J$. Therefore

$$\operatorname{Im} I = -\pi \int_M^{M-\omega} \frac{r_+'^2 + a^2 + l^2}{r_+' - (M-\omega')} \quad d(M-\omega')$$

Finishing the integral we have

$$\operatorname{Im} I = -\pi[(M-\omega)\sqrt{(M-\omega)^2 - a^2 + l^2} + (M-\omega)^2 - M\sqrt{M^2 - a^2 + l^2} - M^2]$$

The tunneling rate

$$\Gamma \sim e^{-2\operatorname{Im} I} = e^{2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 + l^2} - M^2 - M\sqrt{M^2 - a^2 + l^2}]} = e^{\Delta S_{BH}}$$

where ΔS_{BH} is the difference of Bekenstein-Hawking entropy of the blackhole before and after the particle emission from the blackhole.

7.6 Hawking radiation as Tunneling via Hamilton-Jacobi method from Kerr-Newman blackhole: The line element of Kerr-Newmann metric in the Boyer-Lindquist coordinate system is given by,

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - a\,dt]^2$$

.........................(7.6.1)

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$
$$a = \frac{J}{M}$$

Here M is the mass of the body, J is the angular momentum , Q is the electric charge. The equation of the event horizon is given by,

$\Delta = 0$ which gives, $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$, $M^2 > a^2 + Q^2$.

In order to investigate Hawking radiation as tunneling from Kerr-Newmann blackhole we first adopt dragging coordinate system to overcome two difficulties. First , the event horizon $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$ does not coincide with the infinite red-shift surface $r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta - Q^2}$, which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Second , as there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (7.6.1) into a dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ ....(7.6.2)

where Ω is the angular velocity.

For the metric (7.6.1) we have,

$$g_{00} = \frac{-(\Delta - a^2\sin^2\theta)}{\rho^2},\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \frac{\sin^2\theta[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]}{\rho^2}$$

$$g_{03} = \frac{-a\sin^2\theta[(r^2+a^2) - \Delta]}{\rho^2}$$

From (7.6.2), $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}} = \frac{a[(r^2+a^2)-\Delta]}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$ ....(7.6.3)

At the horizon the angular velocity becomes,

$$\Omega_h = \frac{a}{r_+^2 + a^2} \qquad \text{...........................(7.6.4)}$$

The line element (7.6.1) in the dragging coordinate system becomes,

$$ds^2 = \hat{g}_{00}\,dt^2 + g_{11}dt^2 + g_{22}d\theta^2 \qquad \text{...................(7.6.5)}$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2}{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}$.

7.6.1 Tunneling rate of massless particles via Hamilton-Jacobi Method from Kerr-Newman blackhole:

The classical action I of the radiation particle satisfies the relativistic Hamilton-Jacobi equation as [197]

$$g^{\mu\nu}\delta_\mu I \delta_\nu I + u^2 = 0 \qquad \text{..(7.6.6)}$$

where u is the mass of the emitted particle and $g^{\mu\nu}$ are the inverse metric tensor obtained from (7.6.5) as

$$\hat{g}^{00} = -\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Delta\rho^2}, \quad g^{11} = \frac{\Delta}{\rho^2}, \quad g^{22} = \frac{1}{\rho^2}$$

and other components are zero.

Putting these value into the equation (7.6.6) we have

$$\hat{g}^{00}\left(\frac{\delta I}{\delta t}\right)^2 + g^{11}\left(\frac{\delta I}{\delta r}\right)^2 + g^{22}\left(\frac{\delta I}{\delta\theta}\right)^2 + u^2 = 0$$

$$or,\ -\frac{1}{P(r,\theta)}\left(\frac{\delta I}{\delta t}\right)^2 + M(r,\theta)\left(\frac{\delta I}{\delta r}\right)^2 + C(r,\theta)\left(\frac{\delta I}{\delta\theta}\right)^2 + u^2 = 0$$

....(7.6.7)

where,

$$P(r,\theta) = \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Delta\rho^2}, \qquad M(r,\theta) = \frac{\Delta}{\rho^2}, \quad C(r,\theta) = \frac{1}{\rho^2}$$

.................................(7.6.8)

Now considering the axial symmetry of the blackhole , we carry on the following separation variable

$$I = -\omega t + W(r,\theta) + j\phi \qquad \text{...(7.6.9)}$$

where ω is the energy of the emitted particle ,$W(r,\theta)$ is the generalized momentum and j is the angular momentum with respect to the ϕ-axis. From (7.6.9)

$$\frac{\delta I}{\delta t} = -\omega + j\frac{\delta\phi}{\delta t} = -\omega + j\Omega$$

$$\frac{\delta I}{\delta r} = \frac{\delta W}{\delta r}, \frac{\delta I}{\delta\theta} = \frac{\delta W}{\delta\theta} \quad \text{.................................(7.6.10)}$$

Substituting these into equation(7.6.7) we can obtain

$$\frac{\delta W}{\delta r} = \frac{1}{\sqrt{M(r,\theta)\,P(r,\theta)}}\sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W}{\delta\theta}\right)^2 + u^2\right\}}$$

.....................................(7.6.11)

From above equation we can learn that the imaginary part of the emitted particle 's action is only produced from the pole at the event horizon [193]. According to the reference [185] for getting the correct result , the proper spatial distance should be introduced, which is defined by

$$d\sigma^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \quad \text{...(7.6.12)}$$

Since there is no motion in the θ – direction ,so we have

$$\sigma = \int \frac{1}{\sqrt{M(r,\theta)}}dr \quad \text{.............................(7.6.13)}$$

since $M(r,\theta) = \frac{\Delta}{\rho^2}$

Now the horizon

$$\left.\begin{aligned} P(r,\theta) &= P'(r_+,\theta)(r - r_+) + \ldots\ldots \\ M(r,\theta) &= M'(r_+,\theta)(r - r_+) + \ldots\ldots \end{aligned}\right\} \quad \text{...............................(7.6.14)}$$

Where $\frac{\delta P(r,\theta)}{\delta r}\Big|_{r=r_+} = P'(r_+,\theta)$

$\frac{\delta M(r,\theta)}{\delta r}\Big|_{r=r_+} = M'(r_+,\theta)$

Hence $\sigma = \int \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+) + \ldots\ldots}}dr$

$$= \frac{2}{\sqrt{M'(r_+,\theta)}}\sqrt{r - r_+} \quad \text{...(7.6.15)}$$

Again $d\sigma = \frac{dr}{\sqrt{M(r,\theta)}} = \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+) + \ldots\ldots}}dr$

.....................................(7.6.16)

Or, $\frac{d\sigma}{\sigma} = \frac{dr}{2(r - r_+)}$ ....(7.6.17)

From equation (7.6.11) we have

$$\delta W = \frac{2(r-r_+)}{\sqrt{\{M'(r_+,\theta)(r-r_+)+\ldots\ldots\ldots\}\{P'(r_+,\theta)(r-r_+)+\ldots\ldots\ldots\}}}$$

$$\frac{d\sigma}{\sigma}\sqrt{(\omega-j\Omega_+)^2-P(r,\theta)\left\{C(r,\theta)(\frac{\delta W(r,\theta)}{\delta\theta})^2+u^2\right\}}$$

Or, $$W(\sigma)=\frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{d\sigma}{\sigma}\sqrt{(\omega-j\Omega_+)^2-P(r,\theta)\left\{C(r,\theta)(\frac{\delta W(r,\theta)}{\delta\theta})^2+u^2\right\}}$$

....(7.6.18)

where $\Omega_+=\frac{a}{r_+^2+a^2}$ is the velocity at the event horizon and the solution is singular at $\sigma=0$ which corresponds to the event horizon.

Also $\frac{d\sigma}{\sigma}=\frac{dr}{2(r-r_+)}$. Therefore from equation (7.6.18) we can obtain

$$W=\int\frac{2dr}{2(r-r_+)\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\sqrt{(\omega-j\Omega_+)^2-P(r,\theta)\left\{C(r,\theta)(\frac{\delta W}{\delta\theta})^2+u^2\right\}}$$

$$=\frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{dr}{(r-r_+)}\sqrt{(\omega-j\Omega_+)^2-\{P'(r_+,\theta)(r-r_+)+\ldots\ldots\}(0+u^2)}$$

since $\frac{\delta W}{\delta\theta}=0$.

Or, $$W=\frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{(\omega-j\Omega_+)}{(r-r_+)}\sqrt{1-b(r-r_+)}\,dr \quad \text{where } b=\frac{P'(r_+,\theta)u^2}{(\omega-j\Omega_+)^2}$$

$$=\frac{(\omega-j\Omega_+)}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{\sqrt{1-b(r-r_+)}\quad d(r-r_+)}{(r-r_+)}$$

Or, $$W=\frac{(\omega-j\Omega_+)}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\int\frac{\sqrt{1-bz}\quad dz}{z} \quad \text{where } z=r-r_+$$

Here the singularity is at $z=0$. Applying the Cauchy integral formula we can obtain

$$\operatorname{Im}W=\operatorname{Im}I=-\frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(\omega-j\Omega_+)=-\frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(\omega-\frac{ja}{r_+^2+a^2})$$

....(7.6.18)

Now the temperature over the surface of the blackhole is given by[193]

$$T=\frac{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}{4\pi}=\frac{1}{2\pi}\frac{r_+-M}{r_+^2+a^2}$$

.............................(7.6.19)

$$or,\ \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}=\frac{r_+^2+a^2}{2(r_+-M)}$$

From (7.6.18) we have

$$d(\operatorname{Im} I) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}(d\omega - \frac{adj}{r_+^2 + a^2})$$

$$or,\ \operatorname{Im} I = \int \frac{\pi(r_+^2 + a^2)}{r_+ - M}(d\omega - \frac{adj}{r_+^2 + a^2}) \qquad \text{.......................................(7.6.20)}$$

If we fix the total ADM mass and angular momentum of the spacetime and allow those of the blackhole to vary, then when a particle with energy ω and angular momentum j tunnels out, the mass and angular momentum should be modified. Replacing M by $M-\omega$ and J by $J-j$ we obtain the imaginary part of the actual action as

$$\operatorname{Im} I = \pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + a^2}{r_+' - (M-\omega')}(d\omega' - \frac{adj'}{r_+'^2 + a^2})$$

$$= -\pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + a^2}{r_+' - (M-\omega')}\left\{d(M-\omega') - \frac{ad(J-j')}{r_+'^2 + a^2}\right\}$$

$$\text{..(7.6.21)}$$

where

$$J - j' = a(M-\omega'), \qquad r_+ = M + \sqrt{M^2 - a^2 - Q^2}$$

$$r_+' = (M-\omega') + \sqrt{(M-\omega')^2 - a^2 - Q^2}$$

Now, $\int_J^{J-j} d(J-j') = [J-j']_J^{J-j} = (J-j) - J \approx J - J = 0$

j is so small that we assume that $J - j \sim J$. Therefore

$$\operatorname{Im} I = -\pi \int_M^{M-\omega} \frac{r_+'^2 + a^2}{r_+' - (M-\omega')} \, d(M-\omega')$$

Finishing the integral we have

$$\operatorname{Im} I = -\pi[(M-\omega)\sqrt{(M-\omega)^2 - a^2 - Q^2} + (M-\omega)^2 + a^2 \cosh^{-1}\frac{M-\omega}{\sqrt{a^2+Q^2}} - M\sqrt{M^2 - a^2 - Q^2}$$

$$- M^2 - a^2 \cosh^{-1}\frac{M}{\sqrt{a^2+Q^2}}]$$

By comparing ω with M we assume that $a^2 \cosh^{-1}\frac{M-\omega}{\sqrt{a^2+Q^2}} \sim a^2 \cosh^{-1}\frac{M}{\sqrt{a^2+Q^2}}$

Therefore

$$\operatorname{Im} I = -\pi[(M-\omega)\sqrt{(M-\omega)^2 - a^2 - Q^2} + (M-\omega)^2 - M\sqrt{M^2 - a^2 - Q^2} - M^2]$$

The tunneling rate

$$\Gamma \sim e^{-2\,\mathrm{Im}\,I} = e^{2\pi[(M-\omega)^2+(M-\omega)\sqrt{(M-\omega)^2-a^2-Q^2}-M^2-M\sqrt{M^2-a^2-Q^2}]} = e^{\Delta S_{BH}}$$

where ΔS_{BH} is the difference of Bekenstein-Hawking entropy of the blackhole before and after the particle emission from the blackhole.

7.7 Hawking radiation as tunneling via Hamilton-Jacobi method from Kerr-Newman-NUT blackhole: The Kerr-Newman-NUT blackhole metric can be given by [195]

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\rho^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta[r^2+(l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}{\rho^2}d\phi^2 + \frac{2[\Delta(a\sin^2\theta+4l\sin^2\frac{\theta}{2}) - a\sin^2\theta\,[r^2+(l+a)^2]}{\rho^2}dt d\phi \quad \text{.........................(7.7.1)}$$

where

$\Delta = r^2 + a^2 + e^2 + g^2 - l^2 - 2Mr,\qquad \rho^2 = r^2 + (l + a\cos\theta)^2$. Here M is the mass of the blackhole, e and g are the electric and magnetic charges respectively, a is the angular momentum per unit mass, l is the NUT parameter. The event horizon equations are given by $\Delta = 0$ which gives

$$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2 - g^2 + l^2} \quad \text{......................(7.7.2)}$$

The event horizon area of this blackhole is given by [196]

$A = \dfrac{4\pi a}{\Omega_+}$, where Ω_+ is the angular velocity at the horizon.

$$A = 4\pi[r_+^2 + (a+l)^2] \quad \text{.............................(7.7.3)}$$

and Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4} = \pi[r_+^2 + (a+l)^2] = \pi[2M^2 + 2M\sqrt{M^2 - a^2 - e^2 - g^2 + l^2} - e^2 - g^2 + 2l^2 + 2al] \quad \text{.............................(7.7.4)}$$

7.7.1 Dragging coordinate Transformation of Kerr-Newman-NUT blackhole :

The infinite red shift surface is given by $g_{00} = 0$ which gives

$$r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta - e^2 - g^2 + l^2} \quad \text{................................(7.7.5)}$$

Obviously the infinite red shift surface does not coincide with the event horizon surface , which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Also there exist a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (7.7.1) into a dragging coordinate system.

Let $\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}$ ....(7.7.6)

where Ω is the angular velocity.

For the metric (7.7.1) we have,

$$g_{00} = \frac{-(\Delta - a^2 \sin^2\theta)}{\rho^2},\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \frac{\sin^2\theta[(r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}}{\rho^2}$$

$$g_{03} = \frac{\Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}) - a\sin^2\theta[(r^2 + (l+a)^2]}{\rho^2}$$

From (7.7.6),

$$\Omega = \frac{d\phi}{dt} = -\frac{\Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}) - a\sin^2\theta[r^2 + (l+a)^2]}{\sin^2\theta[(r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}$$

..................(7.7.7)

At the horizon the angular velocity becomes,

$$\Omega_+ = \frac{a}{r_+^2 + (l+a)^2} \qquad ....(7.7.8)$$

The line element (7.7.1) in the dragging coordinate system becomes,

$$ds^2 = \hat{g}_{00}\, dt^2 + g_{11} dr^2 + g_{22} d\theta^2 \qquad ....(7.7.9)$$

where $\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{-\Delta\rho^2\sin^2\theta}{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}$.

The line element (7.7.9) represents a 3-dimensional hypersurface of 4-dimensional spacetime. The infinite red-shift surface now coincide with the event horizon surface in the dragging coordinate system. So the geometrical optical limit can be applied now.

7.7.2Tunneling rate of massless particle via Hamilton-Jacobi Method from Kerr-Newman-NUT blackhole :

The classical action I of the radiation particle satisfies the relativistic Hamilton-Jacobi equation as [197]

$$g^{\mu\nu}\delta_\mu I \delta_\nu I + u^2 = 0 \qquad \text{.......................................(7.7.10)}$$

where u is the mass of the emitted particle and $g^{\mu\nu}$ are the inverse metric tensor obtained from (7.7.9) as

$$\hat{g}^{00} = -\frac{\sin^2\theta\left\{r^2 + (l+a)^2\right\}^2 - \Delta(a\sin^2\theta + 4l\sin^2\frac{\vartheta}{2})^2}{\Delta\rho^2\sin^2\theta} \quad , \; g^{11} = \frac{\Delta}{\rho^2} \, , \; g^{22} = \frac{1}{\rho^2}$$

and other components are zero.

From (7.7.10) we obtain ,

$$\hat{g}^{00}(\frac{\delta I}{\delta t})^2 + g^{11}(\frac{\delta I}{\delta r})^2 + g^{22}(\frac{\delta I}{\delta\theta})^2 + u^2 = 0$$

$$-\frac{1}{P(r,\theta)}(\frac{\delta I}{\delta t})^2 + M(r,\theta)(\frac{\delta I}{\delta r})^2 + C(r,\theta)(\frac{\delta I}{\delta\theta})^2 + u^2 = 0 \qquad \text{....................(7.7.11)}$$

where

$$P(r,\theta) = \frac{\left\{r^2 + (l+a)^2\right\}^2\sin^2\theta - \Delta(a\sin^2\theta + 4l\sin^2\frac{\theta}{2})^2}{\Delta\rho^2\sin^2\theta}, \qquad M(r,\theta) = \frac{\Delta}{\rho^2}$$

$$C(r,\theta) = \frac{1}{\rho^2}$$

$$\text{...(7.7.12)}$$

Now , considering the axial symmetry of blackhole spacetime we carry out the separation variable to (7.7.11) as

$$I = -\omega t + W(r,\theta) + j\phi \qquad \text{..(7.7.13)}$$

where ω is the energy of the emitted particle, $W(r,\theta)$ is the generalized momentum and j is the angular momentum with respect to the ϕ - axis. From (7.7.13) we obtain

$$\frac{\delta I}{\delta t} = -\omega + j\frac{\delta\phi}{\delta t} = -\omega + j\Omega, \qquad \frac{\delta I}{\delta r} = \frac{\delta W}{\delta r} \;,\; \frac{\delta I}{\delta\theta} = \frac{\delta W}{\delta\theta} \qquad \text{....................(7.7.14)}$$

Substituting these value into equation (7.7.11) we obtain,

$$\frac{\delta W}{\delta r} = \frac{1}{\sqrt{P(r,\theta)M(r,\theta)}}\sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W}{\delta\theta})^2 + u^2\right\}}$$

$$\text{..................(7.7.15)}$$

where $\Omega = \frac{\delta\phi}{\delta t}$.

From equation (7.7.15) we observe that the imaginary part of the emitted particles action is only produced from the pole at the event horizon[193]. According to the Ref.[185] for getting the correct result the proper spatial distance should be introduced , which is defined by

$$d\sigma^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \quad \text{.............................(7.7.16)}$$

We consider the emitted particle as an ellipsoid shell of energy ω to tunnel across the event horizon and should not have motion in θ-direction ($d\theta = 0$). So we have from equation (7.7.16) ,

$$d\sigma = \frac{1}{\sqrt{M(r,\theta)}} dr$$

$$\sigma = \int \frac{1}{\sqrt{M(r,\theta)}} dr \quad \text{...(7.7.17)}$$

By applying near-horizon approximation we have,

$$\left.\begin{aligned} P(r,\theta) &= P'(r_+,\theta)(r - r_+) + \text{..................} \textit{higher order terms of } (r - r_+) \\ M(r,\theta) &= M'(r_+,\theta)(r - r_+) + \text{..................} \textit{higher order terms of } (r - r_+) \end{aligned}\right\}$$

$$\text{.......................................(7.7.18)}$$

where $\frac{\delta P(r,\theta)}{\delta r}\Big|_{r=r_+} = P'(r_+,\theta)$ *and* $\frac{\delta M(r,\theta)}{\delta r}\Big|_{r=r_+} = M'(r_+,\theta)$

From equation (7.7.17) we obtain

$$\sigma = \int \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+) + \text{..........}}} dr$$

Or, $\sigma = \frac{2}{\sqrt{M'(r_+,\theta)}} \sqrt{(r - r_+)} + \text{..........} \quad \text{...................(7.7.19)}$

Again

$$\begin{aligned} d\sigma &= \frac{1}{\sqrt{M(r,\theta)}} dr \\ &= \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+) + \text{..........}}} dr \\ &= \frac{1}{\sqrt{M'(r_+,\theta)(r - r_+)}} dr \end{aligned}$$

Hence we get $\quad \frac{d\sigma}{\sigma} = \frac{dr}{2(r - r_+)}$

From (7.7.15) we can obtain

$$\delta W = \frac{2(r-r_+)}{\sqrt{\{P'(r_+,\theta)(r-r_+)+........\}\{M'(r_+,\theta)(r-r_+)+........\}}} \frac{d\sigma}{\sigma}$$

$$\sqrt{(\omega - j\Omega_+)^2 - P(r_+,\theta)\left\{C(r_+,\theta)(\frac{\delta W(r_+,\theta)}{\delta\theta})^2 + u^2\right\}}$$

$$W(\sigma) = \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} \int \frac{d\sigma}{\sigma} \sqrt{(\omega - j\Omega_+)^2 - P(r,\theta)\left\{C(r,\theta)(\frac{\delta W(r,\theta)}{\delta\theta})^2 + u^2\right\}}$$

....(7.7.20)

where $\Omega_+ = \frac{a}{r_+^2 + (a+l)^2}$ is the angular velocity at the event horizon, and the solution is singular at $\sigma = 0$ which corresponds to the event horizon. Finishing the integral and substituting the result into (7.7.13) we obtain the imaginary part of the action as

$$\mathrm{Im}\, I = \mathrm{Im}\, W = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\left\{\omega - \frac{ja}{r_+^2 + (l+a)^2}\right\} \quad ....(7.7.21)$$

Now the temperature over the surface of the blackhole is given by [193]

$$T = \frac{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}{4\pi} = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + (a+l)^2}$$

Or, $$\frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} = \frac{r_+^2 + (a+l)^2}{2(r_+ - M)} \quad ....(7.7.22)$$

From (7.7.21) we have, $$d(\mathrm{Im}\, I) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\left\{d\omega - \frac{adj}{r_+^2 + (a+l)^2}\right\}$$

Or, $$\mathrm{Im}\, I = \int \frac{\pi\{r_+^2 + (a+l)^2\}}{r_+ - M}\left\{d\omega - \frac{adj}{r_+^2 + (a+l)^2}\right\}$$

..................................(7.7.23)

If we fix the total ADM mass and angular momentum of the spacetime and allow these of the blackhole to vary, then when a particle with energy ω and angular momentum j tunnels out , the mass and angular momentum should be modified . Replacing M by $M - \omega$ and J by $J - j$ we obtain the imaginary part of the actual action as

$$\mathrm{Im}\, I = \pi \int_M^{M-\omega} \int_J^{J-j} \frac{r_+'^2 + (a+l)^2}{r_+' - (M-\omega')}\left\{d\omega' - \frac{adj'}{r_+'^2 + (a+l)^2}\right\}$$

$$= -\pi \int_M^{M-\omega}\int_J^{J-j} \frac{r_+'^2+(a+l)^2}{r_+'-(M-\omega')}\left\{d(M-\omega') - \frac{ad(J-j')}{r_+'^2+(a+l)^2}\right\} \text{......................(7.7.24)}$$

where $J-j' = a(M-\omega')$, $r_+ = M+\sqrt{M^2-a^2-e^2-g^2+l^2}$ and $r_+' = (M-\omega')+\sqrt{(M-\omega')^2-a^2-e^2-g^2+l^2}$.

Now $\int_J^{J-j} d(J-j') = [J-j']_J^{J-j} = (J-j)-J \approx J-J = 0$

(j is so small that we assume $(J-j)\sim J$)

Hence $\mathrm{Im}\, I = -\pi \int_M^{M-\omega} \frac{r_+'^2+(a+l)^2}{r_+'-(M-\omega')} d(M-\omega')$

$$= -\pi \int_M^{M-\omega} \frac{2(M-\omega')^2+2(M-\omega')\sqrt{(M-\omega')^2-a^2-e^2-g^2+l^2}-e^2-g^2+2l^2+2al}{\sqrt{(M-\omega')-a^2-e^2-g^2+l^2}} d(M-\omega')$$

Finishing the integral we get

$$\mathrm{Im}\, I = -\pi[(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}+(M-\omega)^2-M^2-M\sqrt{M^2-a^2-e^2-g^2+l^2}$$
$$+\frac{1}{2}(a^2-l^2+2al)\cosh^{-1}\frac{(M-\omega)}{\sqrt{a^2+e^2+g^2-l^2}}-\frac{1}{2}(a^2-l^2+2al)\cosh^{-1}\frac{M}{\sqrt{a^2+e^2+g^2}}]$$

$$[\cosh^{-1}\frac{(M-\omega)}{\sqrt{a^2+e^2+g^2-l^2}} \sim \cosh^{-1}\frac{M}{\sqrt{a^2+e^2+g^2-l^2}}]$$

$$\mathrm{Im}\, I = -\pi[(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}+(M-\omega)^2-M^2-M\sqrt{M^2-a^2-e^2-g^2+l^2}]$$

The tunneling rate is therefore

$$\Gamma \sim e^{-2\,\mathrm{Im}\, I} = e^{2\pi[(M-\omega)^2-M^2+(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^{2^2}}-M\sqrt{M^2-a^2-e^2-g^2+l^{2^2}}]}$$

..............................(7.7.25)

Using Bekenstein-Hawking entropy formula $S_{BH} = \pi[r_+^2+(a+l)^2]$,

we have $S_{BH}(M) = \pi[2M^2+2M\sqrt{M^2-a^2-e^2-g^2+l^2}-e^2-g^2+2l^2+2al]$

$$S_{BH}(M-\omega) = \pi[2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}-e^2-g^2+2l^2+2al]$$

$$\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$$
$$= 2\pi[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2} - M^2 - M\sqrt{M^2 - a^2 - e^2 - g^2 + l^2}$$

where ΔS_{BH} is the difference of entropies of the blackhole before and after the emission. From equation (7.7.25) we have

$$\Gamma \sim e^{\Delta S_{BH}} \quad \text{..................................(7.7.26)}$$

7.7.3 Concluding remarks:

In this section, we have presented the Hawking radiation as Hamilton-Jacobi method from the event horizon of Kerr-Newman-NUT blackhole . We find that the emission rate at the event horizon is equal to the difference of Bekenstein-Hawking entropy before and after the emission of a particle.

According to the reference [193] expression (7.7.26) indicates that the radiation is not pure thermal, which gives a correction to the Hawking radiation of the blackhole. Following the reference [193] expanding equation (7.7.26) in terms of $(\omega - \omega_0)$ we get ,

$$\Gamma \sim e^{\Delta S_{BH}} = e^{-\frac{(\omega-\omega_0)}{T}[1-\frac{r_+^2+(a+l)^2}{r_+^4}(M+\sqrt{M^2-a^2-e^2-g^2+l^2}-\frac{M(a^2+e^2+g^2-l^2)}{2(M^2-a^2-e^2-g^2+l^2)}(\omega-\omega_0)+..............]}$$

................................(7.7.27)

When neglecting the higher order terms involving $(\omega - \omega_0)$ the Hawking pure thermal spectrum can be obtained. We therefore come to the conclusion that the actual radiation spectrum of Kerr-Newman-NUT blackhole is not precisely thermal , which provides an interesting correction to Hawking pure thermal spectrum.

In special case , if we put $l = 0$ and we assume the equivalent charge $Q^2 = e^2 + g^2$ then the result is similar for the tunneling of uncharged particle from Kerr-Newman blackhole[18]. If $l = e = g = 0$ then the result reduces to the Kerr blackhole[18] .For $l = a = g = 0$ then the result is fit for Reissner-Nordstrom blackhole and supports the Parikh-Wilezek's result[14]. Also if we assume $a = l = e = g = 0$ then the result is supports for the Schwarzschild blackhole obtained by the Parikh-Wilczek's result [14].

The result we derived above shows that the blackhole radiation causes the spacetime background geometry to be varied. Because of the self-gravitation and energy conservation and angular momentum conservation, the event horizon of blackhole varies with blackhole radiation, namely when the particle outgoes the event horizon will contract and the two turning points pre-contraction and post-contraction are the two points of barrier. The tunneling rate of particle is relevant to the mass M, the angular momentum a,the electric charge e, the magnetic charge g, and the NUT parameters l of the blackhole and satisfies the underlying unitary theory.

References:

[1] J. M. Bardeen , B. Carter and S. W. Hawking ; The Four Laws of Black Hole Mechanics , Commun. math. Phys. 31,161-170 (1973).

[2] R.M.Wald ; Blackholes and Thermodynamics, gr-qc/9702022 , 1997.

[3] J. D. Bekenstein ; Black Holes and Entropy, Phys. Rev. D 7,2333(1973).

[4] S. W. Hawking ; Comm. Math. Phys. 43 (1975) 199.

[5] J. D. Bekenstein ; Lett. Nuov. Cimento 4(1972)737.

[6] R. Penrose and R. M. Floyd ; Nature (Physical Science) 229(1971)177.

[7] D. Christodoulou ; Phys.Rev.Lett.25(1970)1596.

[8] S. W. Hawking ; Phys. Rev. Lett. 26(1971)1344.

[9] S. W. Hawking ; Nature 248, 30 (1974).

[10] S. W. Hawking; Phys. Rev. D14,2460(1976); 72,084013(2005).

[11] Qing-Quan Jiang, Shuang-Qing Wu, Xu- Cai; " Hawking radiation as tunneling from Kerr and Kerr-Newman blackholes" . hep-th/0512351.

[12] J. B. Hartle and S. W. Hawking; Phys. Rev. D13, 2188(1976).

[13] P. Krause and F. Wilezek; Nucl. Phys. B 433(1995) 403, gr-qc/9406042.

[14] M. K. Parikh and F. Wilezek; Phys. Rev. Lett.85,5042(2000) hep-th/9907001.

[15] M. K. Parikh; Phys. Lett. B 546,189(2002) ; hep-th/0204107.

[16] M. K. Parikh; hep-th/ 0402166.

[17] M. K. Parikh; Int. J. Mod. Phys. D 13,2351(2004). [Gen. Rel. Grav. 36, 2419(2004), hep-th/0405160]

[18] J. Zhang, Zheng Zhao; Phys. Lett. B 618(2005)14
[19] W.Liu; Chinese Journal of physics, Vol.45, No.1(2007) February.
[20] S. Sarkar, D.Kothawala; gr-qc/07094448
[21] J. Zhang, Zheng Zhao; gr-qc/0512153.
[22] R, Kerner, R,B. Mann; hep-th/08032246.
[23] Xiao-Xiong Zeng, Hang-Song Hou and Shu-Zheng Yang; PRAMANA, Journal of Physics, Vol. 70, No.3, March(2008).
[24]Ya-Peng Hu, Jing-Yi Zhang, Zheng Zhao; gr-qc/09012680.
[25] R. Kerner, R.B. Mann ; gr-qc/0603019.
[26] U. A. Gillani, M. Rehman and K. Saifullah; hep-th/11020029.
[27] M. Bilal and K.Saifullah; gr-qc/10105575.
[28] M. Rehman and K. Saifullah; hep-th/10115129.
[29] Huei-Chan Lin and Chopin Soo; gr-qc/0905-3244.
[30] M. Arzano, A. J. M. Medved, Elias C. Vagenas; hep-th/0505266
[31] M. Angheben, M. Nadalini , L. vanzo and S. Zerbini; hep-th/0503081.
[32] Yang-Geng Miao, Zhao Xue and Shao-Jun Zhang; hep-th/10122426.
[33] T. Jian, Chan-Bing-Bing; ACTA PHYSICA POLONICA B Vol.40(2009) No.2
[34] B.R.Majhi; hep-th/08091508.
[35] K. Matsuno and K. Umetsu; hep-th/11012091.
[36] M. H.Ali; gr-qc/ 07063890.
[37] M. H.Ali; gr-qc/ 07071079
[38] W. liu; New coordinates of BTZ Black Hole and Hawking radiation via tunneling.
[39] A.J.M. Medved; hep-th/0110289
[40] Shuang-Qing Wu, Qing-Quan Jiang; hep-th/0602033

[41] M. Q. Liu and S.Z. Yang; International Journal of Theoretical Physics, 46(2007)65.

[42]H.L.Li, D.J.Qi, Q.Q.Jiangand S.Z.Yang, International Journal of Theoretical Physics 45(2006)2471

[43] Y. Hu, J. Zhang and Z. Zhao; International Journal of Modern Physics D 16(2007)847; gr-qc/0611085.

[44] Jingyi Zhang and Zheng Zhao ; Hawking radiation of Charged Particles via Tunneling From the Reissner-Nordström blackhole; Published by Institute of Physics Publishing For SISSA, October 2005.

[45]Q.Q.Jiang, S.Z.Yang and S. Q.Wu; International Journal of Theoretical Physics 45(2006)2311

[46] DEYOU CHEN, XIAOTAO ZU; Massive Particles Tunnels From The TAUB-NUT Blackhole; Vol. 39(2008) ACTA PHYSICA POLONICA B No. 6

[47] K. Srinivasan and T. Padmanabhan; Phys. Rev.D60, 024007(1999) ; gr-qc/9812028

[48] S. Shankaranarayann, K. Srinivasan and T. Padmanabhan ; Mod. Phys. Lett. A 16,571(2001); gr-qc/0007022.

[49] S. Shankaranarayann, K. Srinivasan and T. Padmanabhan; Class quant. Grav.19,2671(2002) ; gr-qc/0010042.

[50] S. Shankaranarayann ; Phys. Rev. D 67,084026(2003); gr-qc/0301090.

[51] E. C. Vagenas; Nuovo Cim. B 117,899(2002) ; hep-th/0111047.

[52] Li Gao, Hongbao Zhang and Wenbiao Liu; From Schwarzschild to Kerr due to Black hole Quantum tunneling; Int. J. Theor. Phys. Vol. 46, No.1 January 2007.

[53] Shu-Zheng Yang and Qing-Quan Jiang; Reasearch on Hawking radiation as tunneling from Schwarzschild-anti-de Sitter Black hole;

[54] P.K.Townsend ; Black holes, gr-qc/9707012.

[55] A.Einstein, (1939) On A Stationary System With Spherical Symmetry Consisting of Many Gravitating Masses. (http://www.jstor.org/pss/1968902)

[56] Discovering the Kerr and Kerr-Schild metrics,(http://www.arxiv.org/abs/0706,1109)

[57] T. Damour; The entropy of blackhole-a primer. ar-Xiv : hep-th/0401160.

[58] S.W.Hawking; The brief history of time.

[59] Narit Pidokrajt; Thermodynamics of blackholes

[60] M. Banados, C. Teitelboim and J. Zanelli; The blackhole in three dimensional spacetime.

[61] R.Penrose ; Phys.Rev.Lett. 14, 57-59.

[62] S.W.Hawking; Phys.Rev.Lett. 15,689-6900.

[63] S.W.Hawking; Commun.Math.Phys. 25, 152-166.

[64] D.C.Robinson; Phys.Rev.Lett.34, 905-906.

[65] W.Israel; Phys.Rev. 164,1776(1967)

[66]G.W.Gibbons, S.W.Hawking, G.T.Horowitz and M.J.Perry; Commun.Math.Phys.88,295-308 .

[67] Y.K.Ha ; Int. J. Mod. Phys. D 14,2219-2225.

[68] J. Michell ; 1784 phil,Trans.R.soc(London)74.

[69] A. Celotti, J.C.Miller and D.W.Sciama;
Astrophysical evidence for the existence of black holes.
astro-ph/9912186.
[70] Jean-Pierre Lumient; Black holes: A general introduction.
astro-ph/9801252.
[71] J.Casares; Observational evidence for stellar-mass black holes.
astro –ph/0612312.
[72] S. W. Hawking ; Sci. Am 236(1), 34 (1977).

[73] Barry R. Parker and Robert J. Mcleod ; Black hole thermodynamics in an undergraduate thermodynamic course , Am. J. Phys.48(12) , 1066 (1980).

[74] S. W. Hawking ; The analogy between black hole mechanics and Thermodynamics.
[75] P C W DAVIES ; Thermodynamics of black holes.

[76] J.D.Bekenstein ; Sci.Am., January(1980).

[77] J.D.Bekenstein ; Phys.Rev.D9,3292(1974).

[78] W.G.Unruh and R.M.Wald ; Phys.Rev.D25,942(1982).

[79]W.H.Zurek and K.S.Thorne; Phys.Rev.Lett.54,2171(1985).

[80] V.P.Frolov and D.N.Page; gr-qc/9302017.

[81]W.H.Zurek ; Phys.Rev.Lett.49,1683(1982).

[82]R.M.Wald ; The thermodynamics of blackholes.
Living reviews in relativity,2001.
[83] J.D.Bekenstein ; Phys.Rev.D23,287(1981).

[84] D.N.Page; Hawking radiation and blackhole thermodynamics.
; hep-th/0409024.

[85] K.S.Thorne, W.H.Zurek and R.H.Price; in blackholes: The membrane paradigm, edited by K.S.Thorne, R.H.Price and D.A.Mac Donald
(Yale university press, new Haven,1986)
[86]R.D.Sorkin ; The statistical mechanics of blackholes thermodynamics. Online version; gr-qc/9705006.
[87] R.M.Wald ; Quantum field theory in curved spacetime and blackhole thermodynamics. (University of Chicago press, Chicago,1994).
[88]T.Jacobson ; Introductory lectures on blackhole thermodynamics.

[89] J. A. Wheeler ; A Journey into Gravity and Spacetime, Scientific American Library,
(W.H. Freeman & Co. New York,1990).
[90] J. D. Bekenstein ; Phys. Rev. D 9 (1974)3292.
[91] J. D. Bekenstein ; Do we understand black hole entropy? , gr-qc/9409015.
[92] J. D. Bekenstein ; Physics Today 33(1980) January issue, p.24.
[93] W. Israel, in 300 Years of Gravitation, eds. S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge,1987).

[94] Bekenstein-Hawking entropy ; Scholarpedia , 11/28/2010.
[95] L. Smarr ; Phys. Rev. Lrtt. 30,71(1973).
[96] Parthasarathi Majumder ; Black hole Entropy and Quantum Gravity ; gr-qc/9807045.
[97] D. Christodoulou and R. Ruffini ; Phys. Rev. D 4 , 3552 (1971).
[98] A. Gould ; Classical Derivation Of Black Hole Entropy, Phys. Rev. D 35 , 449 (1987).

[99] Gilad Gour and Avraham E. Mayo, Why is the black hole entropy (almost) linear in the horizon area? gr-qc/0011024.
[100] S. Carlip ; Class. Quat. Grav. 12, 2853(1995).
[101] C. Vaz and L. Witten, The Quantum States and the Statistical Entropy of the Charged Black Hole ; gr-qc/0006039.
[102] V. Frolov and I. Novikov ; Phys. Rev. D, 48,4545(1993); gr-qc/9309001.
[103] L. Bombelli, R. Koul, J. Lee and R. Sorkin ; Phys.Rev.D,34 (1986),373.
[104] Baocheng Zhang, Qing-yu Cai, Ming-Sheng Zhan, Li You ; An interpretation for the entropy of black hole, arXiv:1102.5144 v1 [gr-qc] 25 Feb 2011.

[105] R. M. Wald ; Black hole entropy is Noether charge, arXiv: gr-qc/ 9307038.
[106] Ted Jacobson ; On the nature of the black hole entropy, arXiv: gr-qc/ 9908031.
[107] Jarmo Makela and Pasi Repo ; How to interpret black hole entropy? arXiv: gr-qc/ 9812075.
[108] L. Brillouin ; Science and information theory (Academic, New York,1956).
[109] M. Srednicki ; Phys. Rev. D 71 (1993) 66.
[110] L. Susskind, L. Thorlacius and R. Uglum ; Phys. Rev. D 48 (1993) 3743.
[111] C. Holzhey; Princeton University thesis (unpublished, 1993).

[112] C. Callan and F. Wilczek ; " On Geometric Entropy " , hep-th/9401072.
[113] D. Kabat and D. J.Strassler ; " Coment on Entropy and Area" , hep-th/9401125.
[114] C. Holzhey, F. Larsen and F. Wilczek ; " Geometric and Renormalized Entropy in Conformal Field Theory" , hep-th/9403108.
[115] S. Carlip; Class. Quantum Gravity, 16,3327(1999), gr-qc/9906126.
[116] R.M. Wald ; Phys. Rev .D , 48. 3427(1993), gr-qc/9307038.
[117] G. 't Hooft ; Nuclear Physics B,256, 726(1985); gr-qc/9607022.
[118] M. Bowick , L. Smolin and Wijewardhana ;
L. C. R.; Gen.Rel.Grav. 19,113(1987) .
[119] B. Zwiebach; A first course in string theory,
Cambridge University Press, Cambridge, (2004)
[120] L. Susskind ; Some speculations about blackhole entropy in string theory, unpublished(1993), hep-th/9309145.
[121] A. Strominger and C. Vafa ; Physics Letters B, 379,99(1996), hep-th/9601029.
[122] E. Witten ; Advances in Theoritical and Mathematical Physics, 2,253(1998), ; hep-th/9802150.
[123] V. Frolov ; Blackhole entropy; hep-th/9412211.
[124] V. P. Frolov and D. V. Fursaev ; Plenty of nothing: Blackhole entropy in Induced Gravity , J. Astrophys. Astr. (1999)20, 121-129.
[125] D. V. Fursaev ; Can One Understand Black Hole Entropy Without Knowing Much About Quantum Gravity? ; gr-qc/0404038.
[126] Stefano Liberati ; Problems in Black hole Entropy Interpretation, gr-qc/9601032.

[127] Parthasarathi Majumdar ; Black hole Entropy: Classical and Quantum aspects, hep-th/0110198.
[128] Saurya Das, S. Shankaranarayanan and Sourav Sur ; Black hole entropy from entanglement: A review, gr-qc/0806.0402.
[129] Ted Jacobson, Gungwon Kang and Robert C. Myers ; On Black Hole Entropy , gr-qc/9312023 .

[130] L. Parker, Ph D thesis, Harvard university (1966).

[131] L. Parker; Phys. Rev. Lett. 21562 (1968).

[132] L. Parker ; Phys. Rev.183,1057(1969).

[133] L. Parker, Phys.Rev.D3,346(1971).

[134] S. A. Fulling, Ph D thesis, Princeton university (1972).

[135] S. A. Fulling, Phys. Rev. D7,2850(1973).

[136] N. N. Bogoliubov, V. V. Tolmachev and D. V. Shirkov; A new method in the theory of superconductivity (consultants bureau, New york, 1959)

[137] Ya. B. Zel'dovich ; ZhETF, Pis Red 14,270(1970) English translation in Sov. Phys. JETP Lett. 14,180.

[138]Ya. B. Zel'dovich, Zh, Eksp. Theor. Fiz. 62,2076(1972) English translation in Sov. Phys.JETP Lett. 35,1085

[139] C. W. Misner, Bull. Am. Phys. Soc. 17,472(1972).

[140]A. A. Starobinski, Zh, Eksp. Theor. Fiz.64,48; English translation in Sov. Phys. JETP 37,28(1973)

[141] J. D. Bekenstein ; Phys.Rev.D7,949(1973).

[142] W.G. Unruh; Phys. Rev. Lett. 31,3265(1973)

[143] W. Pauli; Phys. Rev.58,716(1940).

[144] W. G. Unruh; Phys. Rev. D10,3194(1974)

[145] L. H. Ford; Phys. Rev. D12,2963(1975)

[146] D, G, Boulware; Phys. Rev. D13,2169(1976).
[147] P. C. W. Davies; "On the origin of Blackhole evaporation radiation" King's college London preprint(1976).
[148] U. H. Gerlach ; Phys. Rev. D14,1479(1976).
[149] B. S. De Witt, Phys. Rep.19 C ,295 (1975).
[150] W. G, Unruh; Phys. Rev. D14,870(1976).
[151] R. M. Wald; commun. Math. Phys.45,9(1975).
[152] L. Parker; Phys. Rev.D12, 1519(1975).
[153] S. W. Hawking; Phys.Rev. D13,191(1976).
[154] J. D. Bekenstein; Phys. Rev. D 12,3077(1975).

[155] Jarmo Makela ; Black hole spectrum; continuous or discrete? gr-qc/9609001.
[156] J. D. Bekenstein; Lett. Nuovo Cimento 11,467(1974).
[157] Ian I. Kogan; Black hole spectrum, Horizon quantization and all that :(2+1) Dimensional Example: hep-th/9412232.
[158] V. F. Mukhanov, JETP Lett. 44(1986)63
[159] Ya. I. Kogan; JETP Lett. 44(1986)267
[160] J. Garcia-Bellido, Stanford preprint SU-ETP-93-4, hep-th/9302127.
[161] Y. Peleg, Brandeis preprint BRX-TH-350 ; hep-th/9307057
[162] M. Maggiore, Nucl. Phys. B 429(1994)205.
[163] C.O, Lousto; Barselona preprint UAB-FT-340; gr-qc/9405048.
[164] C. O. Lousto and M. Maggiore, Pisa preprint IFUP-TH-53/94 and Barselona preprint , UAB-ET-351, gr-qc/9411037.
[165] A. Astekar, C. Rovelli and L. Smolin; Phys. Rev. Lett. 69(1992)237.
[166] L. Smolin ; Nucl. Phys. b 405(1993)797.

[167] V. A.Berezin, A. M. Boyarsky, A. Ya Neronov; Black hole mass spectrum vs spectrum of Hawking radiation; gr-qc/9808027.

[168] Adam D Helfer; Do black hole radiate? ; gr-qc/0304042.

[169] J. B. Grifiths and J. Podolsky; Class Quantum Grav.22(2005)3467.

[170] J. B. Grifiths and J. Podolsky; Phys. Rev. D 73(2006)044018.

[171] J. F. Plebnski and M. Demianski; Am. Phys. NY 98(1970)98

[172] Usman A. Gillani, Mudassar Rehman and K. Saifullah; hep-th/11020029.

[173] Painleve , P. (1921); Comptes Rendus de l' Academic des Sciences, Serie I(Mathematique) 173,677.

[174]] L. D. Landau, E. M. Lifshitz, the classical theory of field, Pergaman, London(1975).

[175] H. Zhang, Z Zhao, J. Beijing Normal Univ.(Natural Sci.) 37(2001)471(in Chinese).

[176] J. Zhang and Zheng Zhao; JHEP 10(2005)055.

[177] N. Dadhich and Z. Ya. Turakulov; Class Quantum Grav. 19(2002) 2765

[178] Li Hui-Ling, YANG Shu-Zheng and QI De-Jiang; commun. Theor. Phys. (Beijing China) 46(2006) PP.991-994.

[179] J. B. Griffiths and J. Podolosky; gr-qc/0507021.

[180] LEI Jie-Hong, LIU Zhi-Xiang and YANG Shu-Zheng; Commun. Theor. Phys. (Beijing, China)49 (2008) pp. 133-136 , Vol.49 No.1January 15,2008.

[181] Tao Zhu; "Charged" Particle's Tunneling from Rotating Blackholes; hep-th/11014466, 24 January 2011.

[182] K. Umetsu; Tunneling Mechanism in Kerr-Newman Blackhole and Dimensional Reduction near the horizon; Phys. Lett. B 692(2010)61 (hep-th/10071832. Hawking Radiation from Kerr-Newman Blackhole and tunneling mechanism; hep-th/09071420.

K. Matsuno and K. Umetsu; Hawking Radiation as tunneling from squashed Kaluza-Klein Blackhole; hep-th/11012091

[183] S. Iso, H. Umetsu and F. Wilczek; Anomalies, Hawking radiation and regularity in rotating blackholes; Phys. Rev. Lett. 74 (2006)044017 (hep-th / 0606018).

[184] J. Makela and P. Repo; Phys. Rev.D 57, 1899(1998).

[185] P. Krause and E.Keski Vakkuri; Nucl. Phys. B491(1977)219.

R. Parentani Nucl. Phys. B 575(2000) 333.

[186] T. Damour and R. Ruffini; 1996, Phys. Rev.D 14.

[187] M. Angheben, M. Nadalini , L. vanzo and S. Zerbini ;JHEP 05.014(2005).

[188] Liu Jing-Jing, Chen De-You, Yang Shu-Zheng; " A new method to study the Hawking radiation of the charged blackhole with a global monopole" , Rom. Jour. Phys., Vol.53 Nos.5-6, P, 659-664, Bucharest,2008.

[189] Yang-Shu-Zheng, Chen De-You; "A new method to study Hawking radiation of charged particle from stationary axisymmetric Sen Blackhole" ; Chin.Phys. Lett. Vol. 24, No.1

[190] Bo Liu, Xiaokai He, Wenbiao Liu; Coordinate used in derivation of Hawking radiation via Hamilton-Jacobi method; Int. J. Theor. Phys(2009)48; 1334-1340.

[191] Shu Zheng Yang and De You Chen; A New Method to Study Hawking Tunneling Radiation of the Charged Particles from Ressiner-Nordström Black Hole; Int. J. Theor. Phys. Vol. 46, No. 7 July 2007.

[192] R. Kerner and R.B. Mann; Phys. Rev. D 73,104010(2006).

[193]Hong-Lin Liu, Zi-Xiang Liu, Jian-Song Hou, Shu-Zheng Yang; A New Method to Study the Hawking Radiation from the Kerr-NUT Black Hole; Int. J. Theo. Phys.(2008)47, 2960-2965.

[194] N. Dadhich and Z.Ya. Turakulov; Class Quantum Grav.19(2002)2765.

[195] J. B. Griffiths and J. Podolosky; gr-qc/0507021.

[196] V. P. Frolov and I. D. Novikov; Black Hole Physics, Basic Concepts and New Developments; Kluwer Academic Publishers,1998.

[197] Alex Granik; ON AN ELEMENTARY DERIVATION OF THE HAMILTON-JACOBI EQUATION FROM THE SECOND LAW OF NEWTON; arXiv:physics/0309059v4, 16 Oct 2003.

Citations to previously published works.

Part of the contents of this thesis has already appeared in the following papers:

Refereed Journals:

1. M. Ismail Hossain & M. Abdullah Ansary; **"A New Method to Study Hawking Radiation from Kerr-Newman-NUT Black Hole";** Int. J. Theor. Phys. DoI 10.1007/s 10773-012-1264, Vol. 45, No.1 (Published online ;26 July 2012, Springer)

(Chapter- Seven)

2. Md. Ismail Hossain and M. Abdullah Ansary; " **Evidence of Blackhole Entropy"** ; Int. J. Pure Appl. Sci. Technol. 10(1) (2012), pp. 17-26. (Chapter-Four)

3. Md. Ismail Hossain & M. Abdullah Ansary; " **UNCHARGED PARTICLE TUNNELING FROM NON-ACCELERATING AND ROTATING BLACK HOLES WITH ELECTRIC AND MAGNETIC CHARGES".** J.Mech. Cont.& Math. Sci. Vol. 7, No. 1, pp. 957-973, July 2012.

(Chapter-Five)

Accepted for Publication:

4. M. Ismail Hossain & M. Abdullah Ansary; " Some Problems About Blackhole Entropy" . International Journal of Pure and Applied Sciences and Technology. (Chapter- Four)

Appendix:

A New Method to Study Hawking Radiation from Kerr-Newman-NUT Black Hole

Int J Theor Phys
DOI 10.1007/s10773-012-1264-x

A New Method to Study Hawking Radiation from the Kerr-Newman-NUT Black Hole

M. Ismail Hossain · M. Abdullah Ansary

Received: 15 March 2012 / Accepted: 10 July 2012

Abstract Following Hamilton-Jacobi method, we have investigated the Hawking radiation of Kerr-Newman-NUT black hole. We have considered the spacetime background dynamical and incorporate the self-gravitation effect of the emitted particles when energy conservation and angular momentum conservation are taken into account. We have found that the emission rate at the event horizon is equal to the difference of Bekenstein-Hawking entropy before and after emission.

Keywords Hawking radiation · Hamilton-Jacobi method · Kerr-Newman-NUT black hole

1 Introduction

After Hawking's discovery that black holes radiate, there were several approaches to study this effect. The Hawking discovery was based on the general relativity and quantum mechanics. This is the key link in spacetime quantization. In the last few decades, there were many researches on the Hawking radiation and many methods to calculate Hawking radiation were obtained. One of them is the Hamilton-Jacobi method. Our attempt to calculate the tunneling rate of massless particle from the event horizon of Kerr-Newman-NUT black hole by the Hamilton-Jacobi method.

The classical "no hair" theorem states that all the information about the collapsing body is lost except three conserved quantity: the mass, the angular momentum and the electric charge. So the only solutions of Einstein-Maxwell equations in four dimensions is the stationary and rotating Kerr-Newman black hole solutions. In classical theory, the loss of information is not a serious problem since it could be thought that the information is preserved

M.I. Hossain
Department of Mathematics, Kurigram Govt. College, Kurigram, 5600, Bangladesh
e-mail: hossainmath@yahoo.com

M.A. Ansary (✉)
Department of Mathematics, Rajshahi University, Rajshahi 6205, Bangladesh
e-mail: ansary_math@yahoo.com

Published online: 26 July 2012

inside the black hole but just not very accessible. Even, once Hawking thought that the loss of information never recovered. But recently he changed his opinion about information loss paradox. However, taking quantum effect into consideration, the situation is changed due to Hawking discovery that black holes radiates thermally [1, 2].

Due to the emission of thermal radiation black hole could loss energy, shrink and eventually evaporate away completely. Since the radiation with a precisely thermal spectrum carries no information, so the information carried by a physical system falling toward black hole singularity has no away to be recovered after a black hole has disappeared completely. This is known as so called "information loss paradox" [3, 4] which means that pure quantum states can evolve into mixed states. This type of evolution violates the fundamental principle of quantum theory, as these prescribe a unitary time evolution of basis states [5].

The information loss paradox can perhaps be attributed to the semi-classical nature of the investigations of Hawking radiation. However, researches in string theory indeed support the idea that Hawking radiation can be described within a manifestly unitary theory, but it still remains a mystery how information is recovered. Although a complete resolution of the information loss paradox might be within a unitary theory of quantum gravity or string / M-theory, it is argued that the information could come out if the outgoing radiation were not exactly thermal but had subtle corrections [3, 4].

There is some degree of mystery remains in the mechanism of black hole radiation. In the original derivation of black hole evaporations, Hawking described the thermal radiation as a quantum tunneling process created by vacuum fluctuation near the event horizon [6]. In this process, the radiation is like electron-positron pair creation in a constant electric field. The energy of a particle can change its sign after crossing the event horizon. So a pair created by vacuum fluctuations just inside or outside the horizon can materialize with zero total energy, after one member of the pair has tunneled to the opposite side. But Hawking did not proceed in this way. He considered the creation of a black hole in the context of a collapse geometry, calculating the Bogoliubov transformations between the initial and final states of incoming and outgoing radiation. However, there were two difficulties to overcome this problem. The first was to find a well-behaved coordinate system at the event horizon. The second was where is the barrier. Recently, a method to describe Hawking radiation as tunneling process was developed by Krause and Wilezek [7] and elaborated by Parikh and Wilezek [8–12]. This method involves calculating the imaginary part of the action for the process of s-wave emission across the horizon, which in turns is related to the Boltzmann factor for emission at the Hawking temperature. Using the WKB approximation the tunneling probability of the s-wave coming from inside to outside the horizon is given by

$$\Gamma \propto \exp[-2\,\mathrm{Im}\, I], \tag{1}$$

where I is the classical action of the trajectory. Expanding the action in terms of the particle energy, the Hawking temperature is recovered at linear order. In other words for $2I = \beta E + O(E^2)$ this gives

$$\Gamma \sim \exp[-2I] \cong \exp[-\beta E], \tag{2}$$

which is the regular Boltzmann factor for a particle of energy and is the inverse temperature of the horizon.

Besides treating the Hawking radiation as a tunneling process Krause-Parikh-Wilezek also took the tunneling particles back reaction into account. They obtained the corresponding modified spectrum. The most interesting result was that they found this modified spectrum was implicitly consistent with the unitary theory and could support the conservation of information [7–10]. Following this tunneling method, there have been many generalizations, such as its application to other spacetimes. The Hawking radiation as tunneling from various spherically symmetric blackholes were found in [13–28]. Also, there are some attempts to extend this method to the case of stationary axisymmetric black holes [29–37]. Recently, some people investigated the massive charged particles tunneling from the static spherically symmetric as well as stationary axisymmetric black holes [38–45]. They all found a satisfying result. However, Parikh and Wilezek's tunneling method is dependent on coordinates, which means that it should find a Painleve-like coordinates. Recently, Angheben et al. found an invariant tunneling method which was independent of coordinates and called the Hamilton-Jacobi tunneling method to calculate the Hawking temperature [29]. This variant tunneling method could also be considered as an extension of the method used by Padmanabhan et al. [46–50].

The Hamilton-Jacobi method to describe Hawking radiation was developed [51, 52]. In this paper we follow the reference [51, 52] to obtain the tunneling rate of the massless particles at the event horizon of a Kerr-Newman-NUT black hole. The article is arranged as follows. In Sect. 2 we give the metric of Kerr-Newman-NUT black hole. The horizon area and Bekenstein-Hawking entropy formula are also given in this section. In Sect. 3 we introduce the dragging coordinate system in order to infinite red shift surface coincide with the event horizon surface, so that the geometrical optical limit can be applied. In Sect. 4 we discuss the Hamilton-Jacobi process to obtain the tunneling rate. A concluding remarks is given in Sect. 5.

2 Kerr-Newman-NUT Black Hole

The Kerr-Newman-NUT blackhole metric can be given by [50]

$$\begin{aligned} ds^2 = &-\frac{(\Delta - a^2\sin^2\theta)}{\rho^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\sin^2\theta[r^2+(l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\theta/2)^2}{\rho^2}d\phi^2 \\ &+ \frac{2[\Delta(a\sin^2\theta + 4l\sin^2\theta/2) - a\sin^2\theta(r^2+(a^2+l^2))}{\rho^2}dtd\phi, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Delta &= r^2 + a^2 + e^2 + g^2 - l^2 - 2Mr, \\ \rho^2 &= r^2 + (l + a\cos\theta)^2. \end{aligned} \tag{4}$$

Here M is the mass of the black hole. e and g are the electric and magnetic charges respectively, a is the angular momentum per unit mass, l is the NUT parameter. The event horizon equations are given by $\Delta = 0$ which gives

$$r_\pm = M \pm \sqrt{(M^2 - a^2 - e^2 - g^2 + l^2)}. \tag{5}$$

The event horizon area of this black hole is given by [53] $A = \frac{4\pi a}{\Omega}$ where Ω is the angular velocity at the horizon.

$$A = 4\pi\left[r_+^2 + (a+l)^2\right], \tag{6}$$

and Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4} = \pi\left[r_+^2 + (a+l)^2\right]$$
$$= \pi\left[2M^2 + 2M\sqrt{\left(M^2 - a^2 - e^2 - g^2 + l^2\right)} - e^2 - g^2 + 2l^2 + 2al\right]. \qquad (7)$$

3 Dragging Coordinate Transformation

The infinite red shift surface is given by $g_{00} = 0$ which gives

$$r_\pm = M \pm \sqrt{\left(M^2 - a^2\cos^2\theta - e^2 - g^2 + l^2\right)}. \qquad (8)$$

Obviously the infinite red shift surface does not coincide with the event horizon surface, which means that there is an energy layer exists between them. So the geometrical optical limit cannot be applied. Also there exists a frame dragging effect in the stationary rotating spacetime, the matter field in the ergosphere near the horizon must be dragged by the gravitational field also, so a reasonable physical picture should be depicted in the dragging coordinate system. This hints that we must transform the metric (3) into a dragging coordinate system. Let

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{03}}{g_{00}}, \qquad (9)$$

where Ω is the angular velocity. For the metric (3) we have,

$$g_{00} = -\frac{(\Delta - a^2\sin^2\theta)}{\rho^2}, \qquad g_{11} = -\frac{\rho^2}{\Delta}, \qquad g_{22} = \rho^2$$
$$g_{33} = \frac{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\theta/2)^2}{\rho^2} \qquad (10)$$
$$g_{03} = \frac{\Delta(a\sin^2\theta + 4l\sin^2\theta/2) - a\sin^2\theta(r^2 + (a^2 + l^2))}{\rho^2}.$$

From (8),

$$\Omega = \frac{d\phi}{dt} = -\frac{\Delta(a\sin^2\theta + 4l\sin^2\theta/2) - a\sin^2\theta(r^2 + (a^2 + l^2))}{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\theta/2)^2}. \qquad (11)$$

At the horizon the angular velocity becomes,

$$\Omega_+ = \frac{a}{r_+^2 + (l+a)^2}. \qquad (12)$$

The metric (3) in the dragging coordinate system becomes,

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2, \qquad (13)$$

where

$$g^{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{\Delta\rho^2\sin^2\theta}{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\theta/2)^2}. \qquad (14)$$

The line element (13) represents a 3-dimensional hypersurface of 4-dimensional spacetime. The infinite red-shift surface now coincides with the event horizon surface in the dragging coordinate system. So the geometrical optical limit can be applied here.

4 The Hamilton-Jacobi Method

The classical action of the radiation particle satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\delta_\mu I\delta_\nu I + u^2 = 0, \tag{15}$$

where u is the mass of the emitted particle and $g^{\mu\nu}$ are the inverse metric tensor obtained from (13) as

$$g^{00} = \frac{\Delta\rho^2\sin^2\theta}{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\theta/2)^2}, \qquad g^{11} = \frac{\Delta}{\rho^2}, \qquad g^{22} = \frac{1}{\rho^2}, \tag{16}$$

and other components are zero. From (15) we obtain,

$$g^{00}\left(\frac{\delta I}{\delta t}\right)^2 + g^{11}\left(\frac{\delta I}{\delta r}\right)^2 + g^{22}\left(\frac{\delta I}{\delta\theta}\right)^2 + u^2 = 0, \tag{17}$$

$$-\frac{1}{P(r,\theta)}\left(\frac{\delta I}{\delta t}\right)^2 + M(r,\theta)\left(\frac{\delta I}{\delta r}\right)^2 + C(r,\theta)\left(\frac{\delta I}{\delta\theta}\right)^2 + u^2 = 0, \tag{18}$$

where

$$P(r,\theta) = \frac{\Delta\rho^2\sin^2\theta}{\sin^2\theta[r^2 + (l+a)^2]^2 - \Delta(a\sin^2\theta + 4l\sin^2\theta/2)^2},$$
$$M(r,\theta) = \frac{\Delta}{\rho^2}, \qquad C(r,\theta) = \frac{1}{\rho^2}. \tag{19}$$

Now, considering the axial symmetry of black hole spacetime we carry out the separation variable to (13) as

$$I = -\omega t + W(r,\theta) + j\phi, \tag{20}$$

where ω is the energy of the emitted particle, $W(r,\theta)$ is the generalized momentum and j is the angular momentum with respect to the ϕ-axis. From (15) we obtain

$$\frac{\delta I}{\delta t} = -\omega + j\frac{\delta\phi}{\delta t} = -\omega + j\Omega, \qquad \frac{\delta I}{\delta r} = \frac{\delta W}{\delta r}, \qquad \frac{\delta I}{\delta\theta} = \frac{\delta W}{\delta\theta}. \tag{21}$$

Substituting these value into (18) we obtain,

$$\frac{\delta W}{\delta t} = \frac{1}{P(r,\theta)M(r,\theta)}\sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W}{\delta\theta}\right)^2 + u^2\right\}}, \tag{22}$$

where

$$\Omega = \frac{\delta\phi}{\delta t}. \tag{23}$$

From (17) we observe that the imaginary part of the emitted particles action is only produced from the pole at the event horizon. According to the [51] for getting the correct result the proper spatial distance should be introduced, which is defined by

$$d\sigma^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2. \tag{24}$$

We consider the emitted particle as an ellipsoid shell of energy to tunnel across the event horizon and should not have motion in θ-direction ($d\theta = 0$). So we have from (24),

$$d\sigma = \frac{1}{\sqrt{M(r,\theta)}} dr, \tag{25}$$

$$\sigma = \int \frac{1}{\sqrt{M(r,\theta)}} dr. \tag{26}$$

By applying near-horizon approximation we have,

$$\begin{aligned} P(r,\theta) &= P'(r_+,\theta)(r-r_+) + \cdots \text{higher order terms of } (r-r_+), \\ M(r,\theta) &= M'(r_+,\theta)(r-r_+) + \cdots \text{higher orderterms of } (r-r_+), \end{aligned} \tag{27}$$

where

$$\left.\frac{\delta P(r,\theta)}{\delta r}\right|_{r=r_+} = P'(r_+,\theta), \quad \text{and} \quad \left.\frac{\delta M(r,\theta)}{\delta r}\right|_{r=r_+} = M'(r_+,\theta). \tag{28}$$

From (26) we obtain

$$\sigma = \int \frac{1}{\sqrt{M(r_+,\theta)(r-r_+) + \cdots}} dr, \tag{29}$$

implies

$$\sigma = \frac{2}{\sqrt{M'(r_+,\theta)}} \sqrt{(r-r_+)} + \cdots. \tag{30}$$

Again

$$\begin{aligned} d\sigma &= \frac{1}{\sqrt{M(r,\theta)}} dr \\ &= \frac{1}{\sqrt{M(r_+,\theta)(r-r_+) + \cdots}} dr \\ &= \frac{1}{\sqrt{M(r_+,\theta)(r-r_+)}} dr. \end{aligned} \tag{31}$$

Hence we get

$$\frac{d\sigma}{\sigma} = \frac{dr}{2(r-r_+)}. \tag{32}$$

From (22) we can obtain

$$\begin{aligned} \delta W = &- \frac{2(r-r_+)}{\sqrt{\{P'(r_+,\theta)(r-r_+) + \cdots\}\{M'(r_+,\theta)(r-r_+) + \cdots\}}} \frac{d\sigma}{\sigma} \\ &\times \sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W}{\delta\theta}\right)^2 + u^2\right\}}, \end{aligned} \tag{33}$$

$$\begin{aligned} W(\sigma) = &- \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} \\ &\times \int \frac{d\sigma}{\sigma} \sqrt{(\omega - j\Omega)^2 - P(r,\theta)\left\{C(r,\theta)\left(\frac{\delta W}{\delta\theta}\right)^2 + u^2\right\}}, \end{aligned} \tag{34}$$

where $\Omega_+ = \frac{a}{r_+^2 + (a+l)^2}$ is the angular velocity at the event horizon, and the solution is singular at $\sigma = 0$ which corresponds to the event horizon. Finishing the integral and substituting the result into (15) we obtain the imaginary part of the action as

$$\operatorname{Im} I = \operatorname{Im} W = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\left(\omega - \frac{ja}{r_+^2 + (a+l)^2}\right). \tag{35}$$

Now the temperature over the surface of the black hole is given by [48]

$$T = \frac{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}{4\pi} = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + (a+l)^2}, \tag{36}$$

which gives

$$\frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} = \frac{r_+^2 + (a+l)^2}{2(r_+ - M)}. \tag{37}$$

From (23) we have,

$$d(\operatorname{Im} I) = \frac{2\pi}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}\left(d\omega - \frac{adj}{r_+^2 + (a+l)^2}\right). \tag{38}$$

If we fix the total ADM mass and angular momentum of the spacetime and allow these of the black hole to vary, then when a particle with energy ω and angular momentum j tunnels out, the mass and angular momentum should be modified. Replacing M by $M-\omega$ and J by $J-j$ we obtain the imaginary part of the actual action as

$$\begin{aligned}\operatorname{Im} I &= \pi \int_M^{M-\omega}\int_J^{J-j} \frac{r_+'^2 + (a+l)^2}{r_+' - (M-\omega)}\left\{d\omega' - \frac{adj'}{r_+'^2 + (a+l)^2}\right\} \\ &= -\pi \int_M^{M-\omega}\int_J^{J-j} \frac{r_+'^2 + (a+l)^2}{r_+' - (M-\omega)}\left\{d(M-\omega') - \frac{ad(J-j')}{r_+'^2 + (a+l)^2}\right\},\end{aligned} \tag{39}$$

where $J - j' = a(M-\omega')$, $r_+ = M + \sqrt{M^2 - a^2 - e^2 - g^2 + l^2}$ and $r_+' = (M-\omega') + \sqrt{(M-\omega')^2 - a^2 - e^2 - g^2 + l^2}$.

Since j is very small, therefore, we have $\int_J^{J-j} d(J-j') = [J-j']_J^{J-j} = (J-j) - J \approx 0$. Hence

$$\operatorname{Im} I = -\pi \int_M^{M-\omega}\int_J^{J-j} \frac{r_+'^2 + (a+l)^2}{r_+' - (M-\omega)} d(M-\omega'). \tag{40}$$

Using Eq. (39) we have

$$\operatorname{Im} I = -\pi \int_M^{M-\omega}\left[\frac{2(M-\omega')^2 + 2(M-\omega')\sqrt{(M-\omega')^2 - a^2 - e^2 - g^2 + l^2} - e^2 - g^2 + 2l^2 + 2al}{\sqrt{(M-\omega')^2 - a^2 - e^2 - g^2 + l^2}}\right] d(M-\omega'). \tag{41}$$

Finishing the integral we get

$$\begin{aligned}\operatorname{Im} I = -\pi \Bigg[&(M-\omega)\sqrt{(M-\omega)^2 - a^2 - e^2 - g^2 + l^2} + (M-\omega)^2 - M^2 \\ &- M\sqrt{M^2 - a^2 - e^2 - g^2 + l^2} + \frac{1}{2}(a^2 - l^2 + 2al)\cosh^{-1}\frac{(M-\omega)}{\sqrt{a^2 + e^2 + g^2 - l^2}} \\ &- \frac{1}{2}(a^2 - l^2 + 2al)\cosh^{-1}\frac{M}{\sqrt{a^2 + e^2 + g^2 - l^2}}\Bigg],\end{aligned} \tag{42}$$

which gives

$$\begin{aligned}\operatorname{Im} I = -\pi\big[&(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}+(M-\omega)^2-M^2\\&-M\sqrt{M^2-a^2-e^2-g^2+l^2}\big].\end{aligned}\tag{43}$$

Therefore the tunneling rate is

$$\Gamma\sim e^{-2\operatorname{Im} S}=e^{2\pi[(M-\omega)^2-M^2+(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}-M\sqrt{M^2-a^2-e^2-g^2+l^2}]}.\tag{44}$$

Using Bekenstein-Hawking entropy formula $S_{BH}=\pi(r_{\pm}^2+(a+l)^2)$, we have

$$S_{BH}(M)=\pi\big[2M^2+2M\sqrt{M^2-a^2-e^2-g^2+l^2}-e^2-g^2+2l^2+2al\big],\tag{45}$$

and

$$\begin{aligned}S_{BH}(M-\omega')=\pi\big[&2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}\\&-e^2-g^2+2l^2+2al\big].\end{aligned}\tag{46}$$

Therefore

$$\begin{aligned}\Delta S_{BH}&=S_{BH}(M-\omega)-S_{BH}(M)\\&=2\pi\big[(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2-e^2-g^2+l^2}\\&\quad-M^2-M\sqrt{M^2-a^2-e^2-g^2+l^2}\big],\end{aligned}\tag{47}$$

where ΔS_{BH} is the difference of entropies of the black hole before and after the emission. From (44) we have

$$\Gamma\sim e^{\Delta S_{BH}}.\tag{48}$$

5 Concluding Remarks

In this paper, we have used the Hamilton-Jacobi method to presented the Hawking radiation from the event horizon of Kerr-Newman-NUT black hole. We find that the emission rate at the event horizon is equal to the difference of Bekenstein-Hawking entropy before and after the emission of a particle.

According to the reference [51] expression (28) indicates that the radiation is not pure thermal, which gives a correction to the Hawking radiation of the black hole. Following the reference [51] expanding equation (28) in terms of $(\omega-\omega_0)$ we get,

$$\begin{aligned}\Gamma&\sim e^{\Delta S_{BH}}\\&=e^{-\frac{(\omega-\omega_0)}{T}[1-\frac{r_{\pm}^2+(a+l)^2}{r_{\pm}^2}(M+\sqrt{M^2-a^2-e^2-g^2+l^2}-\frac{M(a^2+e^2+g^2-l^2)}{2(M^2-a^2-e^2-g^2+l^2)}(\omega-\omega_0)+\cdots)]}\end{aligned}\tag{49}$$

When neglecting the higher order terms involving $(\omega-\omega_0)$ the Hawking pure thermal spectrum can be obtained. We therefore come to the conclusion that the actual radiation spectrum of Kerr-Newman-NUT black hole is not precisely thermal, which provides an interesting correction to Hawking pure thermal spectrum.

In special case, if we put $l=0$ and we assume the equivalent charge $Q^2=e^2+g^2$ then the result is similar for the tunneling of uncharged particle from Kerr-Newman blackhole [47]. If $l=e=g=0$ then the result reduces to the Kerr black hole [54]. For $l=a=g=0$ then the result is fit for Reissner-Nordstrom black hole and supports the Parikh-Wilczek's result [8]. Also if we assume $a=l=e=g=0$ then the result supports for the Schwarzschild black hole obtained by the Parikh-Wilezek's result [8].

The result we derived above shows that the black hole radiation causes the spacetime background geometry to be varied. Because of the self-gravitation and energy conservation and angular momentum conservation, the event horizon of black hole varies with black hole radiation, namely when the particle outgoes the event horizon will contract and the two turning points pre-contraction and post-contraction are the two points of barrier. The tunneling rate of particle is relevant to the mass, the angular momentum, the electric charge, the magnetic charge, and the NUT parameters of the black hole and satisfies the underlying unitary theory.

References

1. Hawking, S.W.: Nature (London) **248**, 30 (1974)
2. Hawking, S.W.: Commun. Math. Phys. **43**, 199 (1975)
3. Hawking, S.W.: Phys. Rev. D **14**, 2460 (1976)
4. Hawking, S.W.: Phys. Rev. D 72, 084013 (2005)
5. Jiang, Q.-Q., Wu, S.-Q., Cai, X.: Hawking radiation as tunneling from Kerr and Kerr-Newman black-holes. hep-th/0512351
6. Hartle, J.B., Hawking, S.W.: Phys. Rev. D **13**, 2188 (1976)
7. Krause, P., Wilezek, F.: Nucl. Phys. B **433**, 403 (1995). gr-qc/9406042
8. Parikh, M.K., Wilczek, F.: Phys. Rev. Lett. **85**, 5042 (2000). hep-th/9907001
9. Parikh, M.K.: Phys. Lett. B **546**, 189 (2002). hep-th/0204107
10. Parikh, M.K.: hep-th/0402166
11. Parikh, M.K.: Int. J. Mod. Phys. D **13**, 2351 (2004)
12. Parikh, M.K.: Gen. Relativ. Gravit. **36**, 2419 (2004). hep-th/0405160
13. Zhang, J., Zhao, Z.: Phys. Lett. B **618**, 14 (2005)
14. Liu, W.: Chin. J. Phys. **45**(1) (2007)
15. Sarkar, S., Kothawala, D.: 0709.4448 [gr-qc]
16. Zhang, J., Zhao, Z.: Phys. Lett. B **638**, 110 (2006). gr-qc/0512153
17. Jiang, Q.-Q., Wu, S.-Q.: Cai, X.: Phys. Rev. D **73**, 064003 (2006). hep-th/0512351
18. Kerner, R., Mann, R.B.: 0803.2246 [hep-th]
19. Zeng, X.-X., Hou, H.-S., Yang, S.-Z.: PRAMANA, J. Phys. **70**(3), 409–415 (2008)
20. Hu, Y.-P., Zhang, J.-Y., Zhao, Z.: 0901.2680 [gr-qc]
21. Kerner, R., Mann, R.B.: Phys. Rev. D **73**, 104010 (2006). gr-qc/0603019
22. Gillani, U.A., Rehman, M., Saifullah, K.: 1102.0029 [hep-th]
23. Bilal, M., Saifullah, K.: 1010.5575 [gr-qc]
24. Rehman, M., Saifullah, K.: 1011.5129 [hep-th]
25. Lin, H.-C., Soo, C.: 0905.3244 [gr-qc]
26. Arzano, M., Medved, A.J.M., Vagenas, E.C.: J. High Energy Phys. **0509**, 037 (2005). hep-th/0505266
27. Lin, K., Yang, S.-Z.: Phys. Rev. D **79**, 064035 (2009)
28. Yang, S.Z., Lin, K., Yang, J.: Mod. Phys. Lett. A **24**(27), 2187–2193 (2009)
29. Angheben, M., Nadalini, M., Vanzo, L., Zerbini, S.: J. High Energy Phys. **0505**, 014 (2005). hep-th/0503081
30. Miao, Y.-G., Xue, Z., Zhang, S.-J.: 1012.2426 [hep-th]
31. Jian, T., Chan, B.-B.: Acta Physica Polonica B **40**(2) (2009)
32. Majhi, B.R.: 0809.1508 [hep-th]
33. Matsuno, K., Umetsu, K.: 1101.2091 [hep-th]
34. Ali, M.H.: Class. Quantum Gravity **24**, 5849 (2007). 0706.3890 [gr-qc]
35. Ali, M.H.: Int. J. Theor. Phys. **47**, 2203 (2008). 0707.1079 [gr-qc]
36. Liu, W.: New coordinates of BTZ black hole and Hawking radiation via tunneling. Phys. Lett. B **634**, 541–544 (2006)
37. Medved, A.J.M.: Class. Quantum Gravity **19**, 589–598 (2002)
38. Wu, S.-Q., Jiang, Q.-Q.: J. High Energy Phys. **0603**, 079 (2006). hep-th/0602033
39. Griffiths, J.B., Podolsky, J.: Class. Quantum Gravity **22**, 3467 (2005)
40. Griffiths, J.B., Podolsky, J.: Phys. Rev. D **73**, 044018 (2006)
41. Plebnski, J.F., Demianski, M.: Am. Phys. NY **98**, 98 (1970)
42. Gillani, U.A., Rehman, M., Saifullah, K.: 1102.0029 [hep-th]
43. Painleve, P.: C. R. Acad. Sci., Ser. I Math. **173**, 677 (1921)

44. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Field. Pergamon, London (1975)
45. Wang, Q., Yang, S.-Z.: Phys. Scr. **78**, 045003 (2008) (3 pp.)
46. Srinivasan, K., Padmanabhan, T.: Phys. Rev. D **60**, 024007 (1999). gr-qc/9812028
47. Shankaranarayann, S., Srinivasan, K., Padmanabhan, T.: Mod. Phys. Lett. A **16**, 571 (2001). gr-qc/0007022
48. Shankaranarayann, S., Srinivasan, K., Padmanabhan, T.: Class. Quantum Gravity **19**, 2671 (2002). gr-qc/0010042
49. Shankaranarayann, S.: Phys. Rev. D **67**, 084026 (2003). gr-qc/0301090
50. Vagenas, E.C.: Nuovo Cimento B **117**, 899 (2002). hep-th/0111047
51. Liu, H.-L., Liu, Z.-X., Hou, J.-S., Yang, S.-Z.: A new method to study the Hawking radiation from the Kerr-NUT blackhole. Int. J. Theor. Phys. **47**, 2960–2965 (2008)
52. Liu, J.-J., Chen, D.-Y., Yang, S.-Z.: A new method to study the Hawking radiation of the charged black hole with a global monopole. Rom. J. Phys. **53**(5–6), 659–664 (2008)
53. Griffiths, J.B., Podolosky, J.: Class. Quantum Gravity **22**, 3467–3480 (2005). gr-qc/0507021
54. Frolov, V.P., Novikov, I.D.: Black Hole Physics, Basic Concepts and New Developments. Kluwer Academic, Dordrecht (1998)