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Generalizations of Some Properties of Topological and Bitopological Spaces

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Generalizations of Some Properties of Topological and Bitopological Spaces

M.Phil. THESIS

This Thesis is Submitted to the Department of Mathematics, University of Rajshahi for the Degree of Master of Philosophy in Mathematics.

Submitted
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TO

MY BELOVED PARENTS

DECLARATION

I do hereby declare that the whole work submitted as a thesis entitled " **Generalizations of Some Properties of Topological and Bipological Spaces** " to the Department of Mathematics, University of Rajshahi, Bangladesh for the Degree of Master of Philosophy (M.Phil.) in Mathematics is an original research work of mine and have not been previously submitted elsewhere for the award of any other degree.

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CERTIFICATE

This to certify that the thesis entitled "**Generalizations of Some Properties of Topological and Bipological Spaces"** has been prepared by Sanjoy Kumar Biswas under my supervision for submission to the Department of Mathematics, University of Rajshahi, Bangladesh for the Degree of Master of Philosophy (M.Phil.) in Mathematics. It is also certified that the materials include in this thesis are original works of the researcher and have not been previously submitted for the award of any other degree.

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CONTENTS

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Sanjoy Kumar Biswas

ABSTRACT

The thesis is concerned with generalizations of some important and interesting properties of topological and bitopological spaces in a span of four chapters.

The first chapter constitutes an introduction and study of (i) a weak form of strong continuity, (ii) RC- continuity, (iii) perfect continuity, (iv) contra- precontinuity and (v) contra continuity in bitopological spaces . It thus generalizes the corresponding concepts in topology introduced by Donchev, Jafari and Noiri and studied by them. In addition, generalizing the works of Ekici and Noiri, as investigation of relationships between graphs and contra δ-precontinuous functions in bitopological spaces has also been made in this chapter.

In the second chapter the problems of δ-compactness of topological spaces has been generalized to the corresponding properties in bitopological spaces. Some important properties of δ-compactness in bitopological spaces have been established, which are generalizations of results of park, Srivastava and Gupta. Also, a characterization of δ-Hausdorff bitopological spaces has been made and some properties of such spaces have been established, generalizing results of Srivastava and Gupta.

The third chapter introduces the notions of weakly β-continuous functions in tritopological spaces and investigates several properties of these functions, thus generalizing the corresponding works in topological spaces by Khedr, Al-Areefi and Noiri and in bitopological spaces by Tahiliani.

In the fourth chapter the idea of density topology has been introduced for tritopological spaces and has been used to prove certain theorem involving some separation properties. The concept of density of sets in a tritopological spaces and the notion of its trioclosure generalizing topology have been introduced and fruitfully used for study of separation properties.

CHAPTER ONE

ON CONTRA δ-PRECONTINUOUS FUNCTIONS IN BITOPOLOGICAL SPACES

1.1 Introduction

In this chapter, we introduce the notion of contra δ-precontinuous functions in bitopological spaces. Further we obtain a characterization and preservation theorems for contra δ-precontinuous functions in bitopological spaces.

The notion of contra-continuous functions (Donchev 1996) , perfect continuous functions (Noiri 1984a) , contra precontinuous functions (Jafari and Noiri 2002) or RC-continuous functions due to (Donchev and Noiri 1999) plays a significant role in general topology. In this, chapter we introduce and study the notion of weak form of strong continuity , RC-continuity, perfect continuity, contra- precontinuity and contra continuity in bitopological spaces . Also investigated the relationships between graphs and contra δ-precontinuous functions in bitopological spaces , which is a generalization of Ekici and Noiri (2006).

1.2. Preliminaries

In this chapter, the spaces $(X,\mathcal{T}_1,\mathcal{T}_2)$ and $(X,\mathcal{T})$ denote respectively the bitopological space and topological space.

Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space and let A be a subset of X, then the closure and interior of A with respect to $\mathcal{T}i$ are denoted by iCl(A) and iInt(A) respectively , for i = 1,2 .

Definition 1.2.1: A subset A of a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be

(i) (i,j)- regular open (Banerjee (1987)) if A = iInt(jCl(A)) where $i\neq j$, i,j = 1,2.

(ii) (i,j)- regular closed (Bose (1981)) if A = iCl(jInt(A)) where $i\neq j$, i,j = 1,2.

(iii) (i,j)- preopen (Jelic (1990)) if $A \subset$ iInt(jCl(A)) where $i\neq j$, i,j = 1,2.

(iv) (i,j)- semi-open (Bose (1981)) if $A \subset$ jCl(iInt(A)) where $i\neq j$, i,j = 1,2.

Remark 1.2.2: From above definition 1.2.1, we have (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv) but converse are not true. For these we give the following example.

Example 1.2.3: Let X = {a,b,c,d} with topologies $\mathcal{T}_1 = \{X,\phi,\{a\},\{b,c\}\}$, $\mathcal{T}_2 = \{X,\phi,\{b\},\{c,d\}\}$ and A = {c,d} be a subset of X. Then jCl(A) = {a,c,d} and iInt(jCl(A)) = {a}. Therefore iInt(jCl(A)) $\not\subset$ A. Hence (iii) does not imply (i).

Again, let A = {a,b} be a subset of X. Then jInt(A) = {b} and iCl(jInt(A)) = {b,c,d}. Therefore iCl(jInt(A)) $\not\subset$ A. Hence (iv) does not imply (ii).

Definition 1.2.4: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space and A be a subset of X, then

(i) the union of all (i,j)- regular open sets of X contained in A is called (i,j)- δ-interior of a subset A of X and is denoted by (i,j)- δ-(Int(A)) (Velicko 1968).

(ii) A is called (i,j)- δ-open if A = (i,j)- δ-(Int(A)) (Velicko 1968).

(iii) The complement of a (i,j)- δ-open set is called (i,j)- δ-closed . Equivalently, A is (i,j)- δ-closed iff A = (i,j)- δ-(Cl(A)) where (i,j)- δ-(Cl(A)) = $\{x \in X: A \cap U \neq \varphi$, U is (i,j)- δ-open, $x \in U\}$

(iv) A subset A of X is said to be (i,j)- δ-preopen if $A \subset$ iInt((i,j) δ-Cl(A)). The family of all (i,j)- δ-preopen sets of X containing a point $x \in X$ is denoted by (i,j)- δPO(X,x) (M. et al.1982, R and M 1993).

(v) The complement of a (i,j)- δ-preopen set is called (i,j)- δ-preclosed (El-Deeb et al. 1983) .

(vi) The intersection of all (i,j)- δ-preclosed sets of X containing A is called the (i,j)- δ-preclosure of A and is denoted by (i,j)- δ-p(Cl(A)).

(vii) The union of all (i,j)- δ-preopen sets of X contained in A is called the (i,j)- δ-preinterior of A and is denoted by (i,j)- δ-p(Int(A)) (Raychoudhuri and Mukherjee 1993).

(viii) A subset U of X is said to be (i,j)-δ-preneighbourhood (Raychoudhuri and Mukherjee 1993) of a point $x \in X$ if $\exists$ a (i,j)- δ-preopen set V such that $x \in V \subset U$.

(ix) The family of all (i,j)- δ-open (resp. (i,j)- δ-preopen, semi-open, (i,j)- δ-preclosed , (i,j)- closed) sets of X containing a point $x \in X$ is denoted by (i,j)- δO(X,x) (resp. (i,j)- δPO(X,x), (i,j)- SO(X,x) , (i,j)- δPC(X,x), (i,j)-C(X,x)).

Definition 1.2.5: A function f: $(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be

(i) (i,j)-perfect continuous (Noiri 1984 a, N and P. 2007) if $f^{-1}(V)$ is $\mathcal{T}_i$-clopen in X for each σ_i- open set V of Y, for i = 1,2.

(ii) (i,j)-contra-continuous (Dontchev 1996) if $f^{-1}(V)$ is $\mathcal{T}_i$-closed in X for each σ_i- open set V of Y, for i, = 1,2.

(iii) (i,j)-RC- continuous (Dontchev and Noiri 1999) if $f^{-1}(V)$ is (i,j)- regular closed in X for each σ_i- open set V of Y, for $i\neq j$, i,j = 1,2.

(iv) (i,j)-contra-precontinuous (Jafari and Noiri 2002) if $f^{-1}(V)$ is (i,j)- pre-closed in X for each σ_i- open set V of Y, for $i\neq j$, i,j = 1,2.

(v) (i,j)-strongly- continuous (Levine 1960) if $f(iCl(jInt(A))) \subset f(A)$ for every subset A of X .

1.3. Contra δ-Precontinuous Functions in Bitopological Spaces

Definition 1.3.1: A function $f:(X,\mathcal{T}_1,\mathcal{T}_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-contra-δ-precontinuous at a point $x \in X$ if for each σ_i- closed set V in Y with $f(x) \in V$, $\exists$ a (i,j)- δ-preopen set U in X such that $x \in U$ and $f(U) \subset V$ and f is called (i,j)-contra-δ-precontinuous if it has this property at each point of X .

Theorem 1.3.2: The following are equivalent for a function f: $(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$:

(i) f is (i,j)-contra-δ-precontinuous ;

(ii) the inverse image of a σ_{i}- closed set, i = 1,2 of Y is (i,j)- δ-preopen ;

(iii) the inverse image of a σ_{i}- open set, i = 1,2 of Y is (i,j)- δ-preclosed ;

Proof: (i) $\Rightarrow$(ii) . Let V be a σ_{i}- closed set, i = 1,2 in Y with $x \in f^{-1}$(V) . Since f(x) $\in$ V and f is (i,j)-contra-δ-precontinuous , $\exists$ a (i,j)- δ-preopen set U in X containing x such that f(U) $\subset$ V . It follows that $x \in$ U $\subset f^{-1}$(V) . Hence f^{-1}(V) is (i,j)- δ-preopen.

(ii) $\Rightarrow$(iii) . Let U be a σ_{i}- open set, i = 1,2 of Y . Since Y\U is σ_{i}- closed , then by (ii) it follows that f^{-1}(Y\U) = X\ f^{-1}(U) is (i,j)- δ-preopen. Therefore f^{-1}(U) is (i,j)- δ-preclosed in X .

(iii) $\Rightarrow$(i) . Let $x \in$ X and V be a σ_{i}- closed set, i = 1,2 in Y with f(x) $\in$ V . By (iii) , we have f^{-1}(Y\V) = X\ f^{-1}(V) is (i,j)- δ-preclosed and so f^{-1}(V) is (i,j)- δ-preopen . Let U = f^{-1}(V). We obtain that $x \in$ U and f(U) $\subset$ V . This shows that f is (i,j)-contra-δ-precontinuous.

Remark 1.3.3: The following diagram holds:

(i,j)-strongly- continuous

$\Downarrow$

(i,j)-perfect continuous

$\Downarrow$

(i,j)-RC- continuous

$\Downarrow$

(i,j)-contra-continuous

$\Downarrow$

(i,j)-contra-precontinuous

$\Downarrow$

(i,j)-contra- δ -precontinuous

None of these implications are reversible. For these we give the following examples.

Example 1.3.4: Let, X = {a,b,c,d} and $\mathcal{T}_1$ = {X, ϕ ,{a},{b,c}}, $\mathcal{T}_2$ = {X, ϕ ,{b},{c,d}}.

Let, f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(X,$\mathcal{T}_1$,$\mathcal{T}_2$) be the identity function. Then f is (i,j)-perfect continuous but not (i,j)-strongly- continuous. For, let A = {a.b} be a subset of X and f(A) = A, then f(iCl(jInt(A)))$\not\subset$ f(A).

Example 1.3.5: Define the topologies on X = {a,b,c} and Y = {p,q} respectively by $\mathcal{T}_1$ = {X, ϕ ,{b},{a,c}}, $\mathcal{T}_2$ = {X, ϕ ,{a},{b},{a,b},{b,c}} and σ_1 = {Y, ϕ ,{p}}, σ_2 = {Y, ϕ ,{q}}. Let, f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) be a map defined as f(a) = p, f(b) = q, f(c) = p. Then f is (i,j)-RC- continuous but not (i,j)-perfect continuous , since f^{-1}(p) and f^{-1}(q) are clopen in $\mathcal{T}_1$ but not in $\mathcal{T}_2$.

Example 1.3.6: Define the topologies on X = {a, b, c} and Y = {p, q, r} respectively by

$\mathcal{T}_1$ = {X, ϕ, {c}, {b, c}}, $\mathcal{T}_2$ = {X, ϕ, {a}, {c}, {a, c}} and σ_1 = {Y, ϕ, {p}}, σ_2 = {Y, ϕ, {p, q}}. Let, f:(X, $\mathcal{T}_1$, $\mathcal{T}_2$) $\rightarrow$ (Y, σ_1, σ_2) be a map defined as f(a) = p, f(b) = q, f(c) = r. Then f is (i,j)-contra- continuous but not (i,j)-RC- continuous , since then f^{-1}(p, q) is not regular closed in X.

Example 1.3.7: Define the topologies on X = {a, b, c} and Y = {p, q, r} respectively by $\mathcal{T}_1$ = {X, ϕ, {a,b}, {b}}, $\mathcal{T}_2$ = {X, ϕ, {a}, {c}, {a,c}} and σ_1 = {Y, ϕ, {p}}, σ_2 = {Y, ϕ, {r}}. Let, f:(X, $\mathcal{T}_1$, $\mathcal{T}_2$) $\rightarrow$ (Y, σ_1, σ_2) be a map defined as f(a) =p, f(b) = q, f(c) = r. Then f is (i,j)-contra-precontinuous but not (i,j)-contra- continuous, since then f^{-1}(p) is not $\mathcal{T}_i$ -closed in X.

Example 1.3.8: Let ℝ be the set of all real numbers, $\mathcal{P}_2$-be the countable extension topology on ℝ i.e, the topology with subbase $\mathcal{T}_1 \cup \mathcal{T}_2$, where $\mathcal{T}_1$ is the usual topology of ℝ and $\mathcal{T}_2$ is the topology of countable complements of ℝ and σ_1 be the discrete topology of ℝ and $\mathcal{P}_1 = \sigma_2 = \mathcal{T}_1$. Define a function f:(ℝ, $\mathcal{P}_1$, $\mathcal{P}_2$) $\rightarrow$ (ℝ, σ_1, σ_2) as follows

$$f(x) = \begin{cases} 1 \text{ } if \text{ } x \text{ } is \text{ } rational \\ 3 \text{ } if \text{ } x \text{ } is \text{ } irrational \end{cases}$$

Then f is (i,j)-contra-δ-precontinuous but not (i,j)-contra-precontinuous since {1} is closed in ($\mathbb{R}, \sigma_1, \sigma_2$) and $f^{-1}(\{1\}) = \mathbb{Q}$ where $\mathbb{Q}$ is the set of rationals, is not (i,j)-preopen in ($\mathbb{R}, \mathcal{T}_1, \mathcal{T}_2$).

Definition 1.3.9: A function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(Y,σ_1,σ_2) is said to be almost (i,j)-contra-precontinuous (Ekici 2004) if f^{-1}(V) is (i,j)- preclosed in X for each (i,j)- regular open set V in Y.

Remark 1.3.10: Almost contra-precontinuity is a generalization of contra-precontinuity. Almost contra-precontinuity and contra-δ-precontinuity are independent. The following examples prove it.

Example 1.3.11: If we take the function f such as in Example 1.3.4 then f is (i,j)-contra-δ-precontinuous but not almost (i,j)-contra-precontinuous.

Example 1.3.12: Let, X = {a,b,c,d,e}, $\mathcal{T}_1$ = {X,ϕ,{b},{d},{b,d}}, $\mathcal{T}_2$ = {X,ϕ,{a},{c},{a,c}} and Y = {a,b,c,d}, σ_1 = {Y,ϕ,{a},{a,b},{a,c}}, σ_2 = {Y,ϕ,{b},{b,c},{b,d}}. If we take a function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(Y,σ_1,σ_2) defined as f(a) =a, f(b) = b, f(c) = c, f(d) = d, f(e) = d. Then f is almost (i,j)-contra-precontinuous but not (i,j)-contra-δ-precontinuous.

For topological spaces, Noiri and Ekici stated that if A and B be subsets of a space $(X,\mathcal{T})$ and if $A \in \delta PO(X)$ and $B \in \delta O(X)$, then $A \cap B \in \delta PO(B)$ (Raychoudhuri and Mukherjee 1993), then we can state:

Lemma 1.3.13: Let A and B be subsets of a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$. If $A \in (i,j)$-$\delta PO(X)$ and $B \in (i,j)$-$\delta O(X)$, then $A \cap B \in (i,j)$-$\delta PO(B)$.

Proof: We need to prove that $A \cap B \subset$ iInt((i,j)-δ-Cl$(A \cap B)$).

Let, $x \in A \cap B$, then $x \in$ iInt((i,j)-δ-Cl(A)) and $x \in (i,j)$-δ-Int(B), since $A \in (i,j)$-$\delta PO(X)$ and $B \in (i,j)$-$\delta O(X)$. This implies that $\exists$ i-open set G such that, $x \in G \subset (i,j)$-δ-Cl(A).

Also since $x \in (i,j)$-δ-Int(B), this implies that $\exists$ (i,j)-δ-open set U such that $x \in U \subseteq B$ and hence $U \cap A \neq \phi$. Therefore, $\forall$ (i,j)-δ-open set U containing x, $U \cap (A \cap B) \neq \phi$. Hence $x \in G \subset (i,j)$-δ-Cl$(A \cap B)$. Thus $A \cap B \subset$ iInt((i,j)-δ-Cl$(A \cap B)$).

Lemma 1.3.14: Let $A \subset B \subset X$. If $B \in (i,j)$-$\delta O(X)$ and $A \in (i,j)$-$\delta PO(B)$, then $A \in (i,j)$-$\delta PO(X)$ (Raychoudhuri and Mukherjee 1993).

Theorem 1.3.15: If $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is a (i,j)-contra-δ-precontinuous function and A is any (i,j)-δ-open subset of X , then the restriction $f|_A : A \rightarrow Y$ is (i,j)-contra-δ-precontinuous.

Proof: Let F be a σ_{i}- closed set in Y. Then by Theorem 1.3.2 , f^{-1}(F) $\in$ (i,j)-δ PO(X). Since A is (i,j)- δ -open in X , it follows from Lemma 1.3.13 , that $(f|_A)^{-1}$(F) = A $\cap f^{-1}$(F) $\in$ (i,j)-δ PO(A). Hence $f|_A$ is a (i,j)-contra-δ -precontinuous.

Theorem 1.3.16: Let f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(Y,σ_1,σ_2) be a function and $\{U_\alpha : \alpha \in I\}$ be a (i,j)- δ -open cover of X . If for each $\alpha \in I$, f$|_{U_\alpha}$ is (i,j)-contra-δ -precontinuous then f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(Y,σ_1,σ_2) is a (i,j)-contra-δ -precontinuous function.

Proof: Let F be a σ_{i}- closed set in Y. Since for each $\alpha \in I$, f$|_{U_\alpha}$ is (i,j)-contra-δ -precontinuous , $\left(f|_{U_\alpha}\right)^{-1}$ (F) $\in$ (i,j)-δ PO(U_α). Since $U_\alpha \in$ (i,j)-δ O(X), by Lemma 1.3.13 , $\left(f|_{U_\alpha}\right)^{-1}$ (F) $\in$ (i,j)-δ PO(X), for each $\alpha \in I$. Then f^{-1}(F) = $\bigcup_{\alpha \in I}\left[\left(f|_{U_\alpha}\right)^{-1}(F)\right]$ $\in$ (i,j)-δ O(X) . This shows that f is a (i,j)-contra-δ -precontinuous function.

Definition 1.3.17: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space. The collection of all (i,j)-regular open sets forms a base for topology $\mathcal{T}^*$. It is called the semi-regularization. If $\mathcal{T}_1= \mathcal{T}_2=\mathcal{T}^*$ then $(X,\mathcal{T}_1,\mathcal{T}_2)$ is called semi-regular bitopological space.

Theorem 1.3.18: Let $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a function and $g:X\rightarrow X\times Y$ the graph function of f, defined by g(x) = (x,f(x)) for every $x \in X$. If g is (i,j)-contra-δ-precontinuous then f is (i,j)-contra-δ-precontinuous .

Proof: Let U be a σ_i- open set in Y, then $X\times U$ is a σ_i- open set in $X\times Y$. It follows from Theorem 1.3.2 that $f^{-1}(U) = g^{-1}(X\times U)\in$ (i,j)-δPC(X). Thus f is (i,j)-contra-δ-precontinuous .

Lemma 1.3.19: Let A be a subset of a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$. Then $A \in$ (i,j)-δPO(X) iff $A \cap U\in$ (i,j)-δPO(X) for each (i,j)- regular open ((i,j)-δ-open) set U of X (Raychoudhuri and Mukherjee 1993).

Definition 1.3.20: A function $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is called (i,j)-contra-super-continuous for every $x \in X$ and each $F \in$(i,j)-C(Y,f(x)), there exists a (i,j)- regular open set U in X containing x such that $f(U) \subset F$ (Jafari and Noiri 1999).

Definition 1.3.21: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be pairwise Urysohn (Bose and Sinha 1982) if for each distinct points x,y , $\exists$ i - open set U , j-open set V such that $x \in U$, $y \in V$ and $jCl(U) \cap iCl(V) = \phi$ for $i \neq j$, i,j,k=1,2.

Theorem 1.3.22: If $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is (i,j)-contra-super-continuous , $g:X \rightarrow Y$ is (i,j)-contra-δ-precontinuous and Y is pairwise Urysohn , then $E = \{x \in X: f(x) = g(x)\}$ is (i,j)-δ-preclosed in X.

Proof: If $x \in X \setminus E$, then it follows that $f(x) \neq g(x)$. Since Y is pairwise Urysohn , there exist σ_i- open set V and σ_j- open set W such that $f(x) \in V$, $g(x) \in W$ and $jCl(V) \cap iCl(W) = \phi$. Since f is (i,j)-contra-super-continuous and g is (i,j)-contra-δ-precontinuous , there exists a (i,j)- regular open set U containing x and there exists a (i,j)-δ-preopen set G containing x such that $f(U) \subset jCl(V)$ and $g(G) \subset iCl(W)$. Set $O = U \cap G$. By the previous Lemma , O is (i,j)-δ-preopen in X . Hence $f(O) \cap g(O) = \phi$ and it follows that $x \notin$ (i,j)-$\delta PC(E)$. This shows that E is (i,j)-δ-preclosed in X.

Definition 1.3.23: A filter base $\wedge$ is said to be (i,j)-δ-preconvergent (resp. (i,j)-C-convergent) to a point x in X if for any $U \in$ (i,j)-$\delta PO(X)$ containing x (resp. $U \in$(i,j)-C(X) containing x) , there exists a $B \in \wedge$ such that $B \subset U$.

Theorem 1.3.24: If $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is a (i,j)-contra-δ-precontinuous , then for each $x \in X$ and each filter base $\wedge$ in X which is (i,j)-δ-preconvergent to x , the filter base $f(\wedge)$ is (i,j)-C-convergent to f(x) .

Proof: Let $x \in X$ and $\wedge$ be any filter base in X which is (i,j)-δ-preconvergent to x . Since f is (i,j)-contra-δ-precontinuous, then for any $V \in C(Y)$ containing f(x) , there exists $U \in$ (i,j)-δPO(X) containing x such that $f(U) \subset V$. Since $\wedge$ is (i,j)-δ-preconvergent to x there exists a $B \in \wedge$ such that $B \subset U$. It follows that $f(B) \subset V$ and hence the filter base $f(\wedge)$ is (i,j)-C-convergent to f(x) .

Theorem 1.3.25: Let $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a function and $x \in X$. If there exists $U \in$ (i,j)-δO(X) such that $x \in U$ and the restriction of f to U is a (i,j)-contra-δ-precontinuous function at x , then f is (i,j)-contra-δ-precontinuous at x .

Proof: Suppose that $F \in C(Y)$ containing f(x). Since $f|_U$ is (i,j)-contra-δ-precontinuous at x , there exists $V \in$ (i,j)-δPO(U) containing x such that f(V) = $(f|_U)(V) \subset F$. Since $U \in$ (i,j)-δO(X) containing x , it follows from Lemma 1.3.13 that $V \in$ (i,j)-δPO(X) containing x . This shows clearly that f is (i,j)-contra-δ-precontinuous at x .

Definition 1.3.26: A function $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-δ-preirresolute if for each $x \in X$ and each $V \in$ (i,j)-δPO(Y,f(x)), there exists a (i,j)-δ-preopen set U in X containing x such that $f(U) \subset V$.

Theorem 1.3.27: Let $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ and $g:(Y,\sigma_1,\sigma_2) \rightarrow (Z,\Omega_1,\Omega_2)$ be functions. Then the following properties hold :

(i) If f is (i,j)-δ-preirresolute and g is (i,j)-contra-δ-precontinuous , then gof:X$\to$Z is (i,j)-contra-δ-precontinuous .

(ii) If f is (i,j)-contra-δ-precontinuous and g is (i,j)-continuous , then gof:X$\to$Z is (i,j)-contra-δ-precontinuous .

Proof: (i) Let x$\in$X and W$\in$(Z,(gof)(x)) , since g is (i,j)-contra-δ-precontinuous , there exists a (i,j)-δ-preopen set V in Y containing f(x) such that g(V) $\subset$ W . Since f is (i,j)-δ-preirresolute, there exists a (i,j)-δ-preopen set U in X containing x such that f(U) $\subset$ V . This shows that (gof)(U) $\subset$ W . Hence gof is (i,j)-contra-δ-precontinuous .

(ii) Let x $\in$X and W$\in$(Z,(gof)(x)) , since g is (i,j)-continuous , V = g^{-1}(V) is (i,j)-closed. Since f is (i,j)-contra-δ-precontinuous, there exists a (i,j)-δ-preopen set U in X containing x such that f(U) $\subset$ V. Therefore (gof)(U) $\subset$ W . This shows that gof is (i,j)-contra-δ-precontinuous .

Definition 1.3.28: A function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\to$ (Y,σ_1,σ_2) is called (i,j)-δ-preopen if image of each (i,j)-δ-preopen set is (i,j)-δ-preopen .

Theorem 1.3.29: If f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\to$ (Y,σ_1,σ_2) is a surjective (i,j)-δ-preopen function and g:(Y,σ_1,σ_2) $\to$ (Z,Ω_1,Ω_2) is a function such that gof:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\to$ (Z,Ω_1,Ω_2) is (i,j)-contra-δ-precontinuous , then g is (i,j)-contra-δ-precontinuous .

Proof: Let x $\in$X and y $\in$Y such that f(x) = y. Let V$\in$C(Z,(gof)(x)) . Then there exists a (i,j)-δ-preopen set U in X containing x such that g(f(U)) $\subset$ V . Since f is (i,j)-δ-

preopen, f(U) is a (i,j)- δ -preopen set in Y containing y such that g(f(U)) $\subset$ V. This shows that g is (i,j)-contra- δ -precontinuous .

Corollary 1.3.30: Let f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) be a surjective (i,j)- δ -preirresolute and (i,j)- δ -preopen function and let g:(Y,σ_1,σ_2) $\rightarrow$ (Z,Ω_1,Ω_2) be a function . Then gof:X$\rightarrow$Z is (i,j)-contra- δ -precontinuous iff g is (i,j)-contra- δ -precontinuous .

Proof: It can be obtained from Theorem 1.3.24 and Theorem 1.3.26.

Definition 1.3.31: A function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is said to be (i,j)-weakly contra- δ -precontinuous if for each $x \in$X and each σ_i-closed set F, i = 1,2 of Y containing f(x) , $\exists$ a (i,j)- δ -preopen set U in X containing x such that iInt(jClf(U)) $\subset$ V.

Definition 1.3.32: A function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is called (i,j)- δ -pre-semi-open if the image of each (i,j)- δ -preopen set is (i,j)-semi-open .

Theorem 1.3.33: If a function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is (i,j)-weakly contra- δ -precontinuous and (i,j)- δ -pre-semi-open , then f is (i,j)-contra- δ -precontinuous .

Proof: Let $x \in$X and F be a (i,j)-closed set containing f(x) . Since f is (i,j)-weakly contra- δ -precontinuous , $\exists$ a (i,j)- δ -preopen set U in X containing x such that iInt(jCl(f(U))) $\subset$ F. Since f is (i,j)- δ -pre-semiopen , f(U)$\in$(i,j)-SO(Y) and f(U) $\subset$ iCl(jInt(f(U))) $\subset$ F. This shows that f is (i,j)-contra- δ -precontinuous.

1.4. Several Theorems in Bitopological Spaces

In this section, graphs and preservation theorems of (i,j)-contra-δ-precontinuity are studied.

Definition 1.4.1: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be

(i) (i,j)-weakly Hausdorff (Soundararajan , 1971) if each element of X is an intersection of (i,j)-regular closed sets.

(ii) (i,j)-δ-pre-Hausdorff if for each pair of distinct points x and y in X , $\exists$ $U\in$(i,j)-δ PO(X,x) and $V\in$(i,j)-δ PO(X,y) such that $U\cap V = \phi$.

(iii) (i,j)-δ-pre-T_1 if for each pair of distinct points x and y in X, $\exists$ (i,j)-δ-preopen set U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

Here we have given the following examples:

Example 1.4.2: Consider the topologies on X = {a, b, c} be

$\mathcal{T}_1$ = {X, ϕ ,{a},{b}, {c}, {a, b},{b, c},{a,c}} and $\mathcal{T}_2$={X, ϕ ,{a},{b},{a,b},{b,c}}

and let A = {b}, B={b, c}, C={a, c} and D={a, b} be subsets of X, then we have A, B, C, D are (1, 2)-regular closed. Also we have $A\cap B$={b}, $B\cap C$={c} and $C\cap D$={a}. Therefore, X is (1, 2)-weakly Hausdorff.

Example 1.4.3: Consider the topologies on X = {a, b, c} be

$\mathcal{T}_1$ = {X, ϕ ,{a},{b}, {a, b}} and $\mathcal{T}_2$ = {X, ϕ ,{c},{a, c},{b, c}}. Then we have

(1, 2)-δ-preopen sets are X, ϕ , {a},{b},{a, b} and

(2, 1)-δ-preopen sets are X, ϕ, {c},{b, c},{a, c}. Hence (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is a (i,j)-δ-pre-Hausdorff space.

Example 1.4.4: Same as example 1.4.3.

Remark 1.4.5: The following implications are hold for a bitopological space (X,$\mathcal{T}_1$,$\mathcal{T}_2$):

(i) Pairwise $T_1 \Rightarrow$ (i,j)-δ-pre-T_1

(ii) Pairwise $T_2 \Rightarrow$ (i,j)-δ-pre-T_2

These implications are not reversible.

Example 1.4.6: Let X = {a,b,c,d} with topologies $\mathcal{T}_1$ = {X,ϕ,{a},{b,c}}, $\mathcal{T}_2$ = {X,ϕ,{b},{c,d}}. Then (X,$\mathcal{T}_1$,$\mathcal{T}_2$) is (i,j)-δ-pre-T_2 but not pairwise T_2.

Definition 1.4.7: For a function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(Y,σ_1,σ_2), the subset {(x,f(x)): x $\in$X}$\subset$ X$\times$Y is called the graph of f and is denoted by G(f).

Definition 1.4.8: The graph G(f) of a function f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$)$\rightarrow$(Y,σ_1,σ_2) is said to be (i,j)-contra-δ-preclosed if for each (x,y)$\in$(X$\times$Y)\G(f), $\exists$ (i,j)-δ-preopen set U in X containing x and V$\in$(x,y) such that (U$\times$V)$\cap$G(f) = ϕ.

Lemma 1.4.9: The following properties are equivalent for the graph G(f) of a function f :

(i) G(f) is (i,j)-contra-δ-preclosed

(ii) for each (x,y)$\in$(X$\times$Y)\G(f), $\exists$ (i,j)-δ-preopen set U in X containing x and V$\in$(i,j)- (Y,y) such that f(U)$\cap$V = ϕ.

Proof: Obvious.

Theorem 1.4.10: If f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is (i,j)- contra-δ-precontinuous and Y is pairwise Urysohn , G(f) is (i,j)-contra-δ-preclosed in X$\times$Y .

Proof: Suppose that Y is pairwise Urysohn . Let (x,y)$\in$(X$\times$Y)\G(f). It follows that f(x)$\neq$ y. Since Y is pairwise Urysohn , $\exists$ σ_i – open set V and σ_j – open set W such that f(x)$\in$ V, y $\in$ W and jCl(V)$\cap$iCl(W) = ϕ. Since f is (i,j)- contra-δ-precontinuous , $\exists$ (i,j)-δ-preopen set U in X containing x such that f(U) $\subset$ jCl(V). Therefore f(U)$\cap$iCl(W) = ϕ and G(f) is (i,j)-contra-δ-preclosed in X$\times$Y .

Theorem 1.4.11: Let f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) have a (i,j)-contra-δ-preclosed graph. If f is injective, then X is (i,j)-δ-pre-T_1.

Proof: Let x and y be any two distinct points of X. Then we have (x,f(y)) $\in$ (X$\times$Y)\G(f). By Lemma 1.4.9, $\exists$ (i,j)-δ-preopen set U in X containing x and F$\in$C(Y,f(y)) such that f(U) $\cap$ F = ϕ. Hence U $\cap$ f^{-1}(F) = ϕ. Therefore we have y $\notin$ U. This implies that X is (i,j)-δ-pre-T_1.

Definition 1.4.12: A bitopological space (X,$\mathcal{T}_1$,$\mathcal{T}_2$) is called (i,j)-δ-preconnected provided that X is not the union of two disjoint non-empty (i,j)-δ-preopen sets .

Theorem 1.4.13: If f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is (i,j)- contra-δ-precontinuous surjection and X is (i,j)-δ-preconnected , then Y is (i,j)-connected .

Proof: Suppose Y is not (i,j)-connected space . There exist disjoint σ_{i}- open set V_1 and σ_{j}- open set V_2 such that Y = $V_1 \cup V_2$. Therefore V_1 and V_2 are (i,j)-clopen in Y. Since f is (i,j)- contra-δ-precontinuous , $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are (i,j)-δ-preopen in X . Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non-empty disjoint and X = i $f^{-1}(V_1) \cup$ j $f^{-1}(V_2)$. This shows that X is not (i,j)-δ-pre-connected, which is a contradiction. Hence Y is (i,j)-connected .

Definition 1.4.14: A bitopological space (X,$\mathcal{T}_1$,$\mathcal{T}_2$) is called

(i) (i,j)-δ-pre-ultra-connected if every two non-empty (i,j)-δ-preclosed subsets of X intersect ,

(ii) (i,j)-hyperconnected (Steen and Seebach 1970) if every i-open set is j- dense.

Here we have given the following examples:

Example 1.4.15: Consider the topologies on X = {a, b, c} be

$\mathcal{T}_1$ = {X, ϕ ,{a},{b}, {a, b}} and $\mathcal{T}_2$ = {X, ϕ ,{c},{a, c},{b, c}}. Then we have

(1, 2)-δ-preclosed subsets are X, ϕ, {b, c},{a, c},{c} and we see that any two non-empty subsets are intersect, hence (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is (1, 2)-δ-pre-ultra-connected.

Example 1.4.16: Consider the topologies on X = {a, b, c} be

$\mathcal{T}_1$ = {X, ϕ ,{b}, {b, c}} and $\mathcal{T}_2$ = {X, ϕ ,{b},{a, b}}. Then we have

$\mathcal{T}_2$-Cl{b,c} = X and $\mathcal{T}_2$-Cl{b}= X.

Again, $\mathcal{T}_1$-Cl{a, b} = X and $\mathcal{T}_1$-Cl{b} = X.

Hence (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is (i,j)-hyperconnected.

Theorem 1.4.17: If X is (i,j)-δ-pre-ultra-connected and f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is (i,j)- contra-δ-precontinuous and surjective , then Y is (i,j)-hyperconnected.

Proof: Let us suppose that Y is not (i,j)-hyperconnected. Then $\exists$ σ_i – open set V such that V is not j- dense in Y. Then $\exists$ disjoint non-empty σ_i – open subset B_1 and σ_j – open subset B_2 in Y, such that B_1= iInt(jCl(V)) and B_2= Y\jCl(V). Since f is (i,j)- contra-δ-precontinuous and onto, by Theorem 1.3.2, A_1= $f^{-1}(B_1)$ and A_2= $f^{-1}(B_2)$ are disjoint non-empty (i,j)-preclosed subsets of X . By assumption, the (i,j)-δ-pre-ultra-connectedness of X implies that A_1 and A_2 must intersect, which is a contradiction. Hence Y is (i,j)-hyperconnected.

Theorem 1.4.18: If f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is (i,j)- contra-δ-precontinuous injection and Y is pairwise Urysohn , then X is (i,j)-δ-pre-Hausdorff.

Proof: Suppose that Y is pairwise Urysohn. By the injectivity of f , it follows that f(x)$\neq$ f(y) for any distinct points x,y$\in$X. Since Y is pairwise Urysohn , $\exists$ σ_i – open set V and σ_j – open set W such that f(x) $\in$ V, f(y) $\in$ W and jCl(V)$\cap$iCl(W) = ϕ. Since f is (i,j)- contra-δ-precontinuous , $\exists$ (i,j)-δ-preopen set U and G in X containing x and y respectively such that $f(U) \subset jCl(V)$ and $f(G) \subset iCl(W)$. Hence $U \cap G = \phi$. This shows that X is (i,j)- δ -pre-Hausdorff.

Theorem 1.4.19: If $f:(X,\mathcal{T}_1,\mathcal{T}_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is (i,j)- contra- δ -precontinuous injection and Y is (i,j)-weakly Hausdorff then X is (i,j)- δ -pre- T_1 .

Proof: Suppose that Y is (i,j)-weakly Hausdorff. For any distinct points $x,y \in X$, $\exists$ (i,j)-regular closed sets V, W in Y such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is (i,j)- contra- δ -precontinuous, by Theorem 1.3.2, $f^{-1}(V)$ and $f^{-1}(W)$ are (i,j)- δ -preopen subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is (i,j)- δ -pre- T_1 .

Definition 1.4.20: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be

(i) (i,j)- δ -pre-compact (Dontchev 1996) if every (i,j)- δ -preopen (resp. (i,j)-closed) cover of X has a finite subcover

(ii) (i,j)-countably δ -pre-compact ((i,j)-strongly countably S-closed) if every countable cover of X by (i,j)- δ -preopen (resp. (i,j)-closed) sets has a finite subcover.

(iii) (i,j)- δ -pre-Lindelof ((i,j)-strongly S-Lindelof) if every (i,j)- δ -preopen (resp. (i,j)-closed) cover of X has a countable subcover.

Theorem 1.4.21: The (i,j)- contra-δ-precontinuous image of (i,j)-δ-pre-compact ((i,j)- δ-pre-Lindelof, (i,j)-countably δ-pre-compact) space are (i,j)-strongly S-closed (resp.(i,j)-strongly S-Lindelof, (i,j)-strongly countably S-closed).

Proof: Suppose that f:(X,$\mathcal{T}_1$,$\mathcal{T}_2$) $\rightarrow$ (Y,σ_1,σ_2) is (i,j)- contra-δ-precontinuous surjection. Let $\{V_\alpha : \alpha \in I\}$ be any closed cover of Y. Since f is (i,j)- contra-δ-precontinuous, then $\{ f^{-1}(V_\alpha) : \alpha \in I \}$ is a (i,j)-δ-preopen cover of X and hence ∃ a finite subset I_0 of I such that X = $\cup \{ f^{-1}(V_\alpha) : \alpha \in I_0 \}$. Hence we have Y = $\cup \{V_\alpha : \alpha \in I_0\}$ and Y is (i,j)-strongly S-closed.

Similarly, the other proof can be obtained.

Definition 1.4.22: A bitopological space (X,$\mathcal{T}_1$,$\mathcal{T}_2$) is said to be

(i) (i,j)-δ-preclosed-compact if every (i,j)-δ-preclosed cover of X has a finite subcover.

(ii) (i,j)-countably δ-preclosed-compact if every (i,j)-countable cover of X by (i,j)-δ-preclosed sets has a finite subcover.

(iii) (i,j)- δ-preclosed-Lindelof if every cover of X by (i,j)- δ-preclosed set has a countable subcover.

Theorem 1.4.23: The (i,j)- contra-δ-precontinuous image of (i,j)-δ-preclosed-compact ((i,j)- δ-preclosed-Lindelof, (i,j)-countably δ-preclosed-compact) space are pairwise compact (resp. pairwise Lindelof, pairwise countably compact).

Proof: Suppose that f:$(X,\mathcal{T}_1,\mathcal{T}_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)- contra-δ-precontinuous surjection. Let $\{V_\alpha : \alpha \in I\}$ be any open cover of Y . Since f is (i,j)- contra-δ-precontinuous, then $\{ f^{-1}(V_\alpha): \alpha \in I \}$ is a (i,j)-δ-preclosed cover of X. Since X is (i,j)-δ-preclosed-compact , $\exists$ a finite subset I_0 of I such that X = $\cup\{ f^{-1}(V_\alpha)$: $\alpha \in I_0\}$. Hence we have Y = $\cup \{V_\alpha : \alpha \in I_0\}$ and Y is pairwise compact.

Similarly, the other proof can be obtained.

CHAPTER TWO

ON VARIOUS PROPERTIES OF δ-COMPACTNESS IN BITOPOLOGICAL SPACES

2.1. Introduction.

By introducing the notion of δ-compact Anjali Srivastava and Sandhya Gupta in a paper (A. Srivastava and S. Gupta 2005) obtained the generalization of various results of Park in a paper (Herrington and Long 1975 and Park 1988). In this chapter, we introduce the concept on various properties of δ-compactness in bitopological spaces. Jong Suh Park in the paper " H-closed spaces and W-Lindelof spaces " has got various interesting results related with H-closed spaces. Moreover Park has introduced the concept of W-Lindelof spaces which is a generalization of Lindelof spaces. By using the notions of σ-continuous maps, w-closure, w-limit point etc. Park has proved various results concerned with these concepts.

Anjali Srivastava and Sandhya Gupta in the paper " On various properties of δ-compact spaces " have introduced the concept of δ-compact spaces and have got many theorems giving a generalization of Park's theorems by using the tools of δ-continuous maps, w^*-closure, δ-convergence of nets and δ- cluster points of nets etc.

In this chapter, we have introduced the concept of δ-compactness in bitopological spaces and have got many theorems giving a generalization of Park's theorem by using the tools of δ-continuous maps, w^*-closure, δ -convergence of nets and δ- cluster points of nets etc. in bitopological spaces.

In section 2 of this chapter we obtain a characterization of δ-Hausdorff bitopological spaces and discuss various properties of δ-Hausdorff bitopological spaces which compare (i,j)-Hausdorff spaces and Hausdorff spaces. Further the notion of δ-compactness in bitopological spaces is introduced and it is shown that δ-compactness in bitopological spaces is preserved by δ-continuous surjections and w^*-closed sets in bitopological spaces, which is a generalization of A. Srivastava and S. Gupta (2005).

In section 3 of this chapter we study θ-compactness in bitopological spaces a generalization of quasi-H-closed sets and its applications to some forms of continuity using θ-open and δ-open sets in bitopological spaces. Among other results, it is shown that a weakly θ-retract of a Hausdorff spaces X is a δ-closed subset of X in bitopological spaces, which is a generalization of some results of Mohammad Saleh (2004).

2.2. δ-Compactness in Bitopological Spaces.

The section beings with the following definitions of δ-compactness in bitopological spaces.

Definition 2.2.1: (Fletcher et al., 1969). A cover U = $\{U_\alpha \mid \alpha \in A\}$ of $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be a pairwise open cover of X if U $\subset \mathcal{T}_1 \cup \mathcal{T}_2$ and for each $i \in \{1, 2\}$, U $\cap \mathcal{T}_i$ contains a nonempty set, where A is a subset of X.

Definition 2.2.2: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is called (i,j)-δ-compact if for each pairwise open cover $\{U_n\}$ of X there are finitely many n_k such that

$X = \bigcup_{k=1}^{n}$ iInt(jCl(U_{n_k})) where $i \neq j$, i,j =1,2 if U_{n_k} is open in $\mathcal{T}_i$.

Definition 2.2.3: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space. A subset A of X is called a (i,j)-H-closed set or (i,j)-H-set in X (J. Vermeer, 1985) if every pairwise open cover $\{U_n\}$ of X there are finitely many n_k such that $X = \bigcup_{k=1}^{n}$ (jCl(U_{n_k})) where $i \neq j$, i,j =1,2 if U_{n_k} is open in $\mathcal{T}_i$.

Obviously (i,j)-δ-compact space is (i,j)-H-closed. But the converse is not true.

Example 2.2.4: Let X = ℝ, $\mathcal{T}_1$= The usual topology on ℝ, $\mathcal{T}_2$= The discrete topology on ℝ.

Let A = [m, m+r], m, $r \in$ **Z,** r>1 .

Then clearly A is (i,j)-H-closed. Now consider

$\in$ = {(n-1, n)|n$\in$**Z** }$\cup${{s}| s$\in$**Z** }$\cup${the unions of these subsets}. Then $\in$ is a pairwise open cover of A in (X,$\mathcal{T}_1$,$\mathcal{T}_2$).

Let $\in'$ = {(m+r-1, m+r), {{ m+r-1}$\cup${ m+r }}, (m+r-2, m+r-1), {{ m+r-2}$\cup${m+r-1}}, ……., (m, m+1), {{ m}$\cup${ m+1}}}. Therefore,

$\mathcal{T}_2$-Cl(m+i-1, m+i) = (m+i-1, m+i), i = 1,2,….,r-1 and

$\mathcal{T}_1$-Cl{m+i} = {m+i}, i = 1,2,….,r-1.

Clearly, these two types of closures together cover A.

Now, $\mathcal{T}_1$-Int($\mathcal{T}_2$-Cl(m+i-1, m+i)) = (m+i-1, m+i), i= 1,2,….,r-1 and

$\mathcal{T}_2$-Int($\mathcal{T}_1$-Cl({m+i}) = ϕ, i= 1,2,….,r-1. Therefore, the two classes of sets together do not cover A. Hence A is not (i,j)-δ-compact.

Definition 2.2.5: Let (X,$\mathcal{T}_1$,$\mathcal{T}_2$) be a bitopological space. A net (x_n) in X is said to be (i,j)-δ- accumulate to a point x of X denoted by $x_n \overset{\delta}{\infty} x$ if for any i-neighbourhood U of x and n there is an $n_1 \geq$ n such that $x_{n_1} \in$ iInt(jCl(U)) where i$\neq$j, i,j =1,2.

Definition 2.2.6: Let (X,$\mathcal{T}_1$,$\mathcal{T}_2$) be a bitopological space. A net (x_n) in X is said to be (i,j)-δ- converge to a point x of X denoted by $x_n \overset{\delta}{\rightarrow} x$ if for each i-neighbourhood U of x there is an $n_1 \geq$ n such that $x_{n_1} \in$ iInt(jCl(U)) where i$\neq$j, i,j =1,2.

Definition 2.2.7: A net (x_n) on a set X is called universal, or an ultranet (From wikipedia) if for every subset A of X, either (x_n) is eventually in A or (x_n) is eventually in X-A. (By eventually in A we mean, $\exists$ N such that for all $n \geq N$, $x_n \in A$).

Lemma 2.2.8: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space. If a ultranet (x_n) of X (i,j)-δ-accumulate to a point x of X then (x_n) is (i,j)-δ-converge to x.

Definition 2.2.9: If $(X,\mathcal{T}_1,\mathcal{T}_2)$ is a bitopological space, then for any $A \subset X$, we define

(1) $Cl(A) = \cap \{F_1 \cup F_2$ where $A \subset F_1 \cup F_2$ and F_1, F_2 are respectively $\mathcal{T}_1$ and $\mathcal{T}_2$ closed$\}$, then Cl(A) is called a pairwise closure of A.

(2) We also define a pairwise closure in a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ by

$Cl(A) = \{x \in X: A \cap (U \cup V) \neq \phi$, where $A \subset X$ and $x \in U \in \mathcal{T}_1$, $x \in V \in \mathcal{T}_2\}$.

Note that the closure of a subset A w.r.to $\mathcal{T}_1$ and w.r.to $\mathcal{T}_2$ is a subset of the pairwise closure of A.

Note: Now we show that the above two definitions are equivalent.

Proof: Let $x \notin Cl(A)$ in (2). This implies that $A \cap (U \cup V) = \phi \Rightarrow A \cap U = \phi$ and $A \cap V = \phi$. Let U_0 is the union of all $\mathcal{T}_1$ neighborhood U of x and V_0 is the union of all $\mathcal{T}_2$ neighborhood V of x. Then $A \cap U_0 = \phi$ and $A \cap V_0 = \phi$. Therefore $A \subseteq (U_0)^c$

and $A \subseteq (V_0)^c$. Hence $\mathcal{T}_1$-Cl(A) $\subseteq (U_0)^c$ and $\mathcal{T}_2$-Cl(A) $\subseteq (V_0)^c$ where $(U_0)^c$ is closed in $\mathcal{T}_1$ and $(V_0)^c$ is closed in $\mathcal{T}_2$. Therefore $x \notin (U_0)^c \cup (V_0)^c \Rightarrow x \notin Cl(A)$ in (1).

Conversely, let $x \notin Cl(A)$ in (1) then there exist $\mathcal{T}_1$-closed set F_1 and $\mathcal{T}_2$-closed set F_2 such that $x \notin F_1 \cup F_2$ this implies that $x \notin F_1$ and $x \notin F_2$. Therefore $x \in (F_1)^c$ and $x \in (F_2)^c$ implies $x \in (F_1)^c \cup (F_2)^c$ where $(F_1)^c$ is $\mathcal{T}_1$-open and $(F_2)^c$ is $\mathcal{T}_2$-open and $A \cap ((F_1)^c \cup (F_2)^c) = \phi$. Hence $x \notin Cl(A)$ in (2).

Definition 2.2.10: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space. For a subset A of X the (i,j)-weak closure of A denoted by (i,j)-$Cl_w^*(A)$ is defined by the set (i,j)-$Cl_w^*(A) = \{x \in X: A \cap iInt(jCl(U)) \neq \phi$ for all i-open neighborhood U of x $\}$ where $i \neq j$, i,j =1,2.

Example 2.2.11: Consider the topologies on X = {a,b,c} be $\mathcal{T}_1 = \{X, \phi, \{a\}, \{a,c\}\}$ and $\mathcal{T}_2 = \{X, \phi, \{b\}, \{a,b\}\}$. Let $a \in X$ and A = {b,c} be a subset of X, then A is (1,2)-weak closure of A, since for all $\mathcal{T}_1,\mathcal{T}_2$-open neighborhood U of a, we have $A \cap \mathcal{T}_1$-Int($\mathcal{T}_2$-Cl(U)) $\neq \phi$.

Recall that a subset A of X is called (i,j)-regular open if A = iInt(jCl(A)) and X is called (i,j)-semi-regular space if it has a base consisting of (i,j)-regular open sets.

Following lemma establishes the similar behaviour of a pairwise closure and (i,j)-weak closure of a set in terms of the (i,j)-δ-convergence of nets.

Lemma 2.2.12: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space and $A \subseteq X$. Then $x \in$ (i,j)-$Cl_w^*(A)$ iff there is a net (x_n) of points of A, (i,j)-δ-converge to $x \in X$.

Proof: Let $x \in$ (i,j)-$Cl_w^*(A)$. Then $A \cap iInt(jCl(U_n)) \neq \phi$ for all i- neighborhoods U_n of x in X. Consider the family η_x of all i-neighborhood of x with the reverse order inclusion and define a net in X as follows:

$S: \eta_x \to X$ by

$S(U_n) = x_n$ where $x_n \in A \cap iInt(jCl(U_n))$ then (x_n) is a net of point of A and $x_n \xrightarrow{\delta} x$.

Conversely, assume that $x_n \xrightarrow{\delta} x$. For a i- neighborhoods U of x, $\exists\ n_1$ such that $x_n \in iInt(jCl(U))\ \forall\ n \geq n_1$. Since $x_n \in A\ \forall\ n$, we have $A \cap iInt(jCl(U)) \neq \phi$. Thus $x \in$ (i,j)-$Cl_w^*(A)$.

Definition 2.2.13: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space and $A \subseteq X$. Then A is called (i,j)- w^*-closed if $A =$ (i,j)-$Cl_w^*(A)$.

Definition 2.2.14: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space and $A \subset X$. A point x of X is called a (i,j)-δ- limit point of A iff $A \cap \mathcal{T}_2\text{-}Cl(U) \neq \phi$, for every $\mathcal{T}_1$-open set U containing x. The set of all (i,j)-δ- limit points of A is called the (i,j)-δ- closure of A, denoted by (i,j)-δ- Cl(A). A subset A of X is called (i,j)-δ- closed iff A = (i,j)-δ- Cl(A). The complement of (i,j)-δ- closed set is called (i,j)-δ- open.

Definition 2.2.15: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is called (i,j)-δ-Hausdorff if for any two distinict points x and y of X there are i-open neighborhood U of x and j-open neighbourhood V of y such that $iInt(jCl(U))\cap jInt(iCl(V)) = \phi$ where $i\neq j$, i,j =1,2. Equivalently, X is said to be (i,j)-δ- Hausdorff if for every $x\neq y\in X$, $\exists$ (i,j)-δ- open set U_x and (j,i)-δ- open set V_y such that $U_x\cap V_y= \phi$.

Definition 2.2.16: (Noiri and Popa 2007) A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be pairwise Hausdorff if for every x, $y\in X$, $x\neq y$ $\exists$ $U\in\mathcal{T}_1$, $V\in\mathcal{T}_2$ such that $x\in U$, $y\in V$ and $U\cap V = \phi$.

Note that a (i,j)-δ-Hausdorff space is pairwise Hausdorff but the examples can be found which are pairwise Hausdorff but not (i,j)-δ- Hausdorff.

Following theorem gives a characterization of (i,j)-δ-Hausdorff spaces in terms of diagonal of X.

Theorem 2.2.17: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a bitopological space. Then the following statement are equivalent:

(i) X is (i,j)-δ-Hausdorff.

(ii) Every net in X (i,j)-δ-converges to atmost point of X.

(iii) The diagonal $\Delta = \{(x,x):x\in X\}$ is a (i,j)- w^*-closed set of $X\times X$.

Proof: (i)$\Rightarrow$(ii) . Assume that a net (x_n) in X (i,j)-δ-converges to distinict points x and y of X. Since X is (i,j)-δ- Hausdorff there are i-open neighbourhood U of x and j-open neighbourhood V of y such that iInt(jCl(U))$\cap$jInt(iCl(V)) = ϕ . Since $x_n \overset{\delta}{\rightarrow} x$, $\exists$ n_1 such that $x_n \in$ iInt(jCl(U)) $\forall$ $n \geq n_1$. Since $x_n \overset{\delta}{\rightarrow} y$, $\exists$ n_2 such that $x_n \in$ jInt(iCl(V)) $\forall$ $n \geq n_2$.

Choose $m \geq n_1$ and $m \geq n_2$. Then $x_m \in$ iInt(jCl(U))$\cap$jInt(iCl(V)). This is a contradiction. Thus x = y.

(ii)$\Rightarrow$(iii). Let (x,y) $\in$ (i,j)-$Cl_w^*(\Delta)$.Then there is a net (x_n) in X such that $(x_n,x_n) \overset{\delta}{\rightarrow} (x,y)$ .Since $x_n \overset{\delta}{\rightarrow} x$ and $x_n \overset{\delta}{\rightarrow} y$ by (ii) x = y. Thus (x,y) $\in \Delta$.

(iii)$\Rightarrow$(i). Let x,y $\in$ X with x $\neq$ y. Then (x,y) $\notin \Delta$ = (i,j)-$Cl_w^*(\Delta)$. Hence there is a i- neighborhood W of (x,y) such that $\Delta \cap$ iInt(jCl(W)) = ϕ . Choose i-open set U and j-open set V of X with x $\in$ U , y $\in$ V and $U \times V \subset W$. Then iInt(jCl(U))$\cap$jInt(iCl(V)) = ϕ .

Lemma 2.2.18: Let X be a (i,j)-δ - compact space. Then for each net (x_n) in X there is an x $\in$ X such that $x_n \overset{\delta}{\infty} x$.

Proof: Suppose that (x_n) has no (i,j)- δ - limit point in X. Then (x_n) is not (i,j)-δ - accumulate to a point x in X. For each x $\in$ X there is a i-neighborhood U_x of x and n_x such that $x_n \notin$ iInt(jCl(U_x)) $\forall$ $n \geq n_x$. Then {U_x: x $\in$ X } is a pairwise open cover of

X. Since X is (i,j)- δ - compact , there are finitely many x_k such that $X = \bigcup_{k=1}^{n}$ iInt(jCl(U_{x_k})). Choose m such that $m \geq n_{x_k}$ $\forall$ k = 1,2,…..,n. Conclude from above $x_m \notin \bigcup_{k=1}^{n}$ iInt(jCl(U_{x_k})) $\forall$ k = 1,2,…..,n. This contradiction shows that (x_n) has necessarily a (i,j)- δ - cluster point in X.

Theorem 2.2.19: If a bitopological space X is (i,j)- δ - compact then every net in X has a (i,j)- δ -convergent subnet.

Proof: Let (x_n) be a net in X. Since every net has a ultra subnet, (x_n) has a ultra subnet (x_{n_k}). Then by above lemma 2.2.17 there is an $x \in X$ such that $x_{n_k} \overset{\delta}{\infty} x$. Therefore we have $x_{n_k} \overset{\delta}{\rightarrow} x$.

Theorem 2.2.20: Let X be a (i,j)- δ - compact space. If A is (i,j)-Cl_w^*-closed subset of X, then A is (i,j)- δ - compact.

Proof: Let (x_n) be a net in A. Then (x_n) is a net in X. Since X is (i,j)- δ - compact (x_n) has a (i,j)- δ -convergent subnet. Let $x_n \overset{\delta}{\rightarrow} x$. Since $x \in$ (i,j)-Cl_w^*(A) and A is a (i,j)-Cl_w^*-closed we conclude that $x \in A$. It shows that A is (i,j)- δ - compact.

Theorem 2.2.21: Let X be a (i,j)- δ - Hausdorff space. Then every (i,j)- δ - compact subset of X is (i,j)- w^*-closed.

Proof: Let $x \in$ (i,j)-$Cl_w^*(A)$. Then there is a net (x_n) in A such that $x_n \xrightarrow{\delta} x$. Then x is a (i,j)-$\delta$- limit point of (x_n). Since A is (i,j)-δ- compact, $x \in A$. Hence (i,j)-$Cl_w^*(A) = A$ i.e, A is a (i,j)- w^*-closed set of X.

Definition 2.2.22: A function f: $(X,\mathcal{T}_1,\mathcal{T}_2) \to (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-δ- continuous at a point x if for each σ_i –neighborhood U of f(x) there are $\mathcal{T}_i$-neighborhood V of x such that $f(iInt(jCl(V))) \subset iInt(jCl(U))$ where $i \neq j$, i,j = 1,2. If f is (i,j)-δ- continuous at every $x \in X$, then f is called (i,j)-δ- continuous.

Definition 2.2.23: (Noiri, Khedr and AL-Areefi,1992) A mapping f: $(X,\mathcal{T}_1,\mathcal{T}_2) \to (Y,\sigma_1,\sigma_2)$ is said to be pairwise continuous if inverse image of every σ_1-open (resp. σ_2-open) set in Y is $\mathcal{T}_1$-open (resp. $\mathcal{T}_2$ -open) in X.

Note that the concepts of pairwise continuous maps and (i,j)-δ- continuous maps are different.

Example 2.2.24: Define the topologies on X = {a, b, c} and Y = {p, q, r} respectively by $\mathcal{T}_1 = \{X,\phi,\{a,b\},\{a\}\}$, $\mathcal{T}_2 = \{X,\phi,\{b\},\{b,c\}\}$ and $\sigma_1 = \{Y,\phi,\{p\},\{p,r\}\}$, $\sigma_2 = \{Y,\phi,\{q\}\}$. Let, $f:(X,\mathcal{T}_1,\mathcal{T}_2) \to (Y,\sigma_1,\sigma_2)$ be a map defined as f(a) = p, f(b) = q, f(c) = r. Then f is (1,2)-δ- continuous since if $a \in X$ and σ_1- neighborhood U = {p,r} we have $\mathcal{T}_1$-neighborhood V = {a} such that $f(\mathcal{T}_1\text{-}Int(\mathcal{T}_2\text{-}Cl(V))) \subset \sigma_1\text{-}Int(\sigma_2\text{-}Cl(U))$.

But it is not a pairwise continuous maps since inverse image of σ_1– open set $f^{-1}(p,r)= \{a,c\}$ in Y which is not $\mathcal{T}_1$ –open in X.

Following theorem gives a characterization of (i,j)- δ - continuous maps between two spaces.

Theorem 2.2.25: A mapping $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is (i,j)- δ - continuous at $x\in X$ iff for any net (x_n) in X satisfying $x_n \xrightarrow{\delta} x$, the net $f((x_n)) \xrightarrow{\delta} f(x)$ in Y.

Proof: Given any σ_i- neighborhood U of f(x), there is a $\mathcal{T}_i$-neighborhood V of x such that $f(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(V))) \subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}(U))$ where $i\neq j$, i,j = 1,2.

Also there is an n_1 such that $x_n \in i\text{Int}(j\text{Cl}(V))$ for all $n \geq n_1$. Since $f(x_n) \in f(i\text{Int}(j\text{Cl}(V))) \subset i\text{Int}(j\text{Cl}(U))$ for all $n \geq n_1$, we have $f((x_n)) \xrightarrow{\delta} f(x)$.

Conversely, assume that f is not (i,j)- δ - continuous at x. Then there is a σ_i- neighborhood U of f(x) such that $f(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(V))) \not\subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}(U))$ where $i\neq j$, i,j = 1,2 for all $\mathcal{T}_i$-neighborhood V of x. Let (V_n) be the family of $\mathcal{T}_i$-neighborhoods of x with the reverse inclusion order. For each n, since $f(i\text{Int}(j\text{Cl}(V_n))) \not\subset i\text{Int}(j\text{Cl}(U))$, there is an $x_n \in i\text{Int}(j\text{Cl}(V_n))$ such that $f(x_n) \notin i\text{Int}(j\text{Cl}(U))$. Then the net (x_n) in X (i,j)- δ - converges to x but the net $f(x_n)$ in Y does not (i,j)- δ - converges to f(x). Thus we have a contradiction. Hence f is (i,j)- δ - continuous at x.

Definition 2.2.26: A mapping $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to have (i,j)-w^*-closed graph if its graph $G(f) = \{x,f(x): x\in X\}$ is (i,j)-w^*-closed subset of $X\times Y$.

Theorem 2.2.27: A mapping $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ has a (i,j)-w^*-closed graph iff for any net (x_n) in X ,$x_n\overset{\delta}{\rightarrow}x\in X$ and $f(x_n)\overset{\delta}{\rightarrow}y\in Y$ implies $y = f(x)$.

Proof: Assume that $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ has a (i,j)-w^*-closed graph. Since $(x_n,f(x_n))$ is a net in G(f) and $(x_n,f(x_n))\overset{\delta}{\rightarrow}(x,y)$, we have $(x,y)\in$ (i,j)-$Cl_w^*(G(f)) = G(f)$. Thus $y = f(x)$.

Conversely, assume that $(x,y)\in$ (i,j)-$Cl_w^*(G(f))$. Then there is a net (x_n) in X such that $(x_n,f(x_n))\overset{\delta}{\rightarrow}(x,y)$. Since $x_n\overset{\delta}{\rightarrow}x$ and $f(x_n)\overset{\delta}{\rightarrow}$ f(x) , $y = f(x)$.Thus $(x,y)\in G(f)$. Hence G(f) is (i,j)-w^*-closed.

Theorem 2.2.28: Let (Y,σ_1,σ_2) be a (i,j)-δ - Hausdorff space. Then every (i,j)-δ - continuous mapping $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ has a (i,j)-w^*-closed graph.

Proof: Let $(x,y)\in$ (i,j)-$Cl_w^*(G(f))$. Then there is a net (x_n) in X such that $(x_n,f(x_n))\overset{\delta}{\rightarrow}(x,y)$. Then $x_n\overset{\delta}{\rightarrow}x$ and $f(x_n)\overset{\delta}{\rightarrow}y$. Since f is (i,j)-$\delta$ - continuous at x, $f(x_n)\overset{\delta}{\rightarrow}f(x)$. Since Y is (i,j)-$\delta$ - Hausdorff, $y = f(x)$. This implies $(x,y)\in G(f)$. Hence G(f) is (i,j)-w^*-closed.

Theorem 2.2.29: Let (Y,σ_1,σ_2) be a (i,j)-δ- compact space. If a mapping $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ has a (i,j)-w^*-closed graph then f is (i,j)-δ- continuous.

Proof: Let (x_n) be a net in X and $x_n\xrightarrow{\delta}x$. Since Y is (i,j)-δ- compact the net $(f(x_n))$ in Y has a (i,j)-δ- convergent subnet by Theorem 2.2.17. Let $f(x_n)\xrightarrow{\delta}y\in Y$. Since $(x_n,f(x_n))\xrightarrow{\delta}(x,y)$, $(x,y)\in$ (i,j)-$Cl_w^*(G(f)) = G(f)$. Thus y = f(x) and so $f(x_n)\xrightarrow{\delta}f(x)$. This means that f is (i,j)-δ- continuous at x.

Theorem 2.2.30: Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a (i,j)-δ- compact space and (Y,σ_1,σ_2) a bitopological space. If $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is (i,j)-δ- continuous surjection, then Y is a (i,j)-δ- compact.

Proof: Let (y_n) be a net in Y. For each n, there is an $x_n\in$ X such that $y_n = f(x_n)$. Since X is (i,j)-δ- compact, there is a subnet (x_{n_k}) of (x_n) and an $x\in X$ such that $x_{n_k}\xrightarrow{\delta}x$. Since f is (i,j)-$\delta$- continuous at x, $f(x_{n_k})\xrightarrow{\delta}f(x)$. Thus Y is (i,j)-$\delta$- compact.

2.3. Hausdorffness and Weak Forms of Compactness in Bitopological Spaces.

Definition 2.3.1: Recall that in chapter one a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be pairwise Urysohn (Bose and Sinha 1982) if for each distinct points x,y , $\exists$ i - open set U , j-open set V such that $x \in U$, $y \in V$ and $jCl(U) \cap iCl(V) = \phi$ for $i \neq j$, i,j =1,2.

Lemma 2.3.2: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is (i,j)-R-Hausdorff if for every $x \neq y \in X$, $\exists$ (i,j)-regular open set U_x and (j,i)-regular open set V_y such that $U_x \cap V_y = \phi$.

By a (i,j)- weak θ- restriction we mean a (i,j)- weak θ- continuous function $f:(X,\mathcal{T}_1,\mathcal{T}_2) \to A$ where $A \subset X$ and f|A is the identity function on A. In this case A is said to be a (i,j)- weak θ- restriction of X.

The next theorem is an improvement of Theorem 3.3 of (M. Saleh 2004).

Theorem 2.3.3: Let $A \subset X$ and $f:(X,\mathcal{T}_1,\mathcal{T}_2) \to A$ be a (i,j)- weak θ- restriction of X onto A . If X (i,j)-R-Hausdorff , then A is an (i,j)- δ - closed subset of X.

Proof: Suppose not, then there exists a point $x \in$ (i,j)- δ - Cl(A). Since f is a (i,j)- weak θ- restriction, we have $f(x) \neq x$. Since X is (i,j)-R-Hausdorff, there exist $\exists$ (i,j)-regular open set U and (j,i)-regular open set V of x and f(x) respectively such that $U \cap V = \phi$. Let W be any open set in X containing x. Then $U \cap iInt(jCl(W))$ is a (i,j)-regular open set containing x and hence $iInt(jCl(U)) \cap iInt(jCl(W)) \cap A \neq \phi$, since $x \in$ (i,j)- δ - Cl(A). Therefore, $\exists$ a point $y \in iInt(jCl(U)) \cap iInt(jCl(W)) \cap A$. Since $y \in A$, $f(y) = y \in iInt(jCl(U))$ and hence $f(y) \in jCl(V)$. This shows that f(iInt(jCl(W))) is not contained in jCl(V). This contradicts the hypothesis that f is a (i,j)- weak θ- continuous . Thus A is a (i,j)- δ - closed as claimed.

Definition 2.3.4: A function f:$(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j)- weak θ- continuous at $x\in X$ if given any σ_i-open set V in Y containing f(x) , $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that f(iInt(jCl(V)))$\subset$jCl(U) where $i\neq j$, i,j = 1,2. If this condition is satisfied at each point $x\in X$, then f is said to be (i,j)- weak θ- continuous (briefly, (i,j)-w. θ.c) .

Theorem 2.3.5: Let f, g be (i,j)-w. θ.c from a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ into a pairwise Urysohn space (Y,σ_1,σ_2). Then the set A = {$x\in X$: f(x) = g(x)} is an an (i,j)- δ - closed set.

Proof: We will show that X\A is (i,j)- δ - open. Let $x\in A^c$. Then $f(x)\neq g(x)$. Since Y is a pairwise Urysohn, $\exists$ σ_i-open set $W_{f(x)}$ and σ_j-open set $V_{g(x)}$ such that jCl(W)$\cap$iCl(V) = ϕ. By (i,j)-w. θ.c of f and g , $\exists$ (i,j)-regular open set U_1 and (j,i)-regular open set U_2 of x such that $f(U_1)\subset$jCl(W) and $g(U_2)\subset$ iCl(V). Clearly U = $U_1\cap U_2\subset$ X\A. Thus X\A is (i,j)- δ - open and hence A is (i,j)- δ - closed.

Definition 2.3.6: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space, then $A\subset X$ is called (i,j)- θ- dense if (i,j)- $Cl_\theta(A) = X$.

Corollary 2.3.7: Let f, g be (i,j)-w. θ.c from a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ into a pairwise Urysohn space (Y,σ_1,σ_2). If f and g agree on a (i,j)- θ- dense subset of X then f = g every where.

Theorem 2.3.8: Let $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be (i,j)-w. θ.c map and let $A\subset X$. Then $f:A\rightarrow Y$ is (i,j)-w. θ.c.

Proof: Straight forward.

Remark 2.3.9: If $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is (i,j)-w. θ.c map. Then $f:X\rightarrow f(X)$ need not be (i,j)-w. θ.c.

Definition 2.3.10: A function $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j)- almost strongly-θ- continuous at $x\in X$ if given any σ_i-open set V in Y containing f(x) , $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that $f(jCl(U))\subset jInt(iCl(V))$ where $i\neq j$, i,j = 1,2. If this condition is satisfied at each point $x\in X$, then f is said to be (i,j)- almost strongly-θ- continuous (briefly, (i,j)-a.**s.**c) .

Definition 2.3.11: A function $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j)- weak continuous at $x\in X$ if given any σ_i-open set V in Y containing f(x) , $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that $f(U)\subset iCl(V)$ where $i\neq j$, i,j = 1,2. If this condition is satisfied at each point $x\in X$, then f is said to be (i,j)- weak continuous (briefly, (i,j)-w.c) .

Definition 2.3.12: A function $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j)- δ-continuous at $x\in X$ if given any σ_i-open set V in Y containing f(x) , $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that $f(iInt(jCl(U)))\subset iInt(jCl(V))$ where $i\neq j$, i,j = 1,2. If this condition is satisfied at each point $x\in X$, then f is said to be (i,j)- δ-continuous (briefly, (i,j)- δ.c) .

Definition 2.3.13: A subset A of a bitopological space X is said to be (i,j)- θ_ compact if every cover of (i,j)- θ_open sets has a finite subcover.

Lemma 2.3.14: A subset A of a bitopological space X is (i,j)- δ_compact iff every cover of (i,j)- δ _open sets has a finite subcover.

Theorem 2.3.15: Let $f:(X,\mathcal{T}_1,\mathcal{T}_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be (i,j)-w. θ.c and K be an (i,j)- δ_ compact subset of X. Then f(K) is a (i,j)- δ-compact subset of Y.

Proof: Let V be an open cover of f(K). For each $k\in K$, $f(k)\in v_k$ for some $v_k\in V$. By (i,j)-w. θ.c of f , $f^{-1}(Cl(v_k))$ is (i,j)-regular open. The collection $\{ f^{-1}(Cl(v_k)) : k\in K\}$ is a (i,j)-regular open cover of K and so since K is (i,j)- δ_compact , there is a finite subcollection $\{ f^{-1}(Cl(v_k)) : k\in v_o\}$ where v_o is a finite subset of K and $\{ f^{-1}(Cl(v_k)) : k\in v_o\}$ covers K. Clearly, $\{ (Cl(v_k)) : k\in v_o\}$ covers f(K) and thus f(K) is a (i,j)- δ-compact subset of Y.

Theorem 2.3.16: A (i,j)- δ_compact subset of a (i,j)- δ_Hausdorff space is (i,j)- δ_ closed.

Proof: Let A be a (i,j)- δ -compact subset of a (i,j)- δ -Hausdorff space X. We will show that X\A is (i,j)- δ -open. Let, $x\in$ X\A then for each $a\in A$, $\exists$ (i,j)- δ -open set $U_{x,a}$ and (j,i)- δ -open set V_a such that $U_{x,a}\cap V_a = \phi$. The collection $\{v_a: a\in A\}$ is a (i,j)- δ -open cover of A. Therefore, $\exists$ a finite subcollection $v_1, v_2, \ldots, v_n$ that covers A. Let U = $U_1\cap \ldots\ldots\cap U_n$, then $U\cap A = \phi$. Thus X\A is (i,j)- δ -open , proving that A is (i,j)- δ -closed.

Theorem 2.3.17: Every (i,j)- δ -closed subset of a (i,j)- δ -compact space is (i,j)- δ -compact.

Proof: Let X be a (i,j)- δ -compact and let A be a (i,j)- δ -closed subset of X. Let C be a (i,j)- δ -open cover of A, then C plus X\A is a (i,j)- δ -open cover of X. Since X is (i,j)- δ -compact, this collection has a finite subcollection that covers X. But then C has a finite subcollection that covers A as we need.

Theorem 2.3.18: A (i,j)- δ -compact subset of a (i,j)- θ -Hausdorff space is (i,j)- θ -closed.

Proof: Let A be a (i,j)- δ -compact subset of a (i,j)- θ -Hausdorff space X. We will show that X\A is (i,j)- θ -open. Let, $x\in$ X\A then for each $a\in A$, $\exists$ (i,j)- θ -open set $U_{x,a}$ and (j,i)- θ -open set V_a such that $U_{x,a}\cap V_a = \phi$. The collection $\{v_a: a\in A\}$ is a (i,j)- θ -open cover of A. Therefore, $\exists$ a finite subcollection $v_1, v_2, \ldots, v_n$ that covers A. Let

$U = U_1 \cap \ldots\ldots\ldots \cap U_n$, then $U \cap A = \phi$. Thus X\A is (i,j)- θ-open. , proving that A is (i,j)- θ-closed.

Definition 2.3.19: A bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be pairwise connected (Previn 1967) if it can not be expressed as the union of two non empty disjoint sets U and V such that U is i-open and V is j-open , where $i \neq j$, i,j = 1,2.

Theorem 2.3.20: Let $f:(X,\mathcal{T}_1,\mathcal{T}_2) \to (Y,\sigma_1,\sigma_2)$ be a surjective (i,j)-w. θ.c and let X be pairwise connected. Then Y is pairwise connected.

Proof: Suppose Y is pairwise disconnected. Then $\exists$ σ_i-open set V and σ_j-open set W such that $Y = V \cup W$. By (i,j)-w. θ.c of f, $f^{-1}(jCl(V)) = f^{-1}(V)$ and $f^{-1}(iCl(W)) = f^{-1}(W)$ are open in X. But $X = f^{-1}(V) \cup f^{-1}(W)$ and $f^{-1}(V) \cap f^{-1}(W) = \phi$. Thus X is pairwise disconnected, a contradiction. Therefore, Y is pairwise connected.

CHAPTER THREE

A NOTE ON WEAKLY β-CONTINUOUS FUNCTIONS IN TRITOPOLOGICAL SPACES

3.1 Introduction

As a generalization of β-continuous functions and weakly β-continuous functions in bitopological space, we introduce and study some properties of weakly β-continuous functions in tritopological spaces and we obtain its some characterizations.

In general topology the notation of semi-preopen sets due to Andrijevic (1986) or β-open sets due to Mashhour et al (1983) plays a significant role. In Mashhour et al (1983) the concept of β-continuous functions is introduced and further Popa and Noiri (1994) studied the concept of weakly β-continuous functions. In 1992, Khedr et al. introduced and studied β-continuity in bitopological spaces. Recently 2008, Sanjay Tahiliani introduced and studied weakly β-continuous functions in bitopological spaces. In this chapter, we introduce and study the notation of weakly β-continuous functions in tritopological spaces and investigate several properties of these functions in tritopological spaces.

The concept of tritopological spaces are introduced in (Hassan (2014)).

3.2 Basic Definitions

In the present chapter, the space $(X,\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$, $(X,\mathcal{P}_1,\mathcal{P}_2)$, and $(X,\mathcal{T})$ denote the tritopological, bitopological and topological spaces respectively.

Let $(X,\mathcal{T})$ be a topological space and A be a subset of X. The closure and interior of A are denoted by Cl(A) and Int(A) respectively.

In $(X,\mathcal{P}_1,\mathcal{P}_2)$ the closure and interior of $A \subseteq X$ w. r. to $\mathcal{P}_i$ are denoted by iCl(A) and iInt(A) respectively, for i=1,2.

In $(X,\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$, the closure and interior of $A \subseteq X$ w. r. to $\mathcal{P}_i$ are denoted by iCl(A) and iInt(A) respectively, for i=1,2,3.

Definition 3.2.1: (Hassan (2014)) Let X be a non-empty set. If $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ are three collections of subsets of X such that $(X, \mathcal{P}_1)$, $(X, \mathcal{P}_2)$ and $(X, \mathcal{P}_3)$ are three topological spaces then X is called a tritopological space and is denoted by $(X,\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$.

Example 3.2.2: Let $X = \{a, b, c, d\}$

$\mathcal{P}_1 = \{X, \phi, \{c, d\}\}$, $\mathcal{P}_2 = \{X, \phi, \{a, b, c\}, \{b\}\}$ and $\mathcal{P}_3 = \{X, \phi, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$

then $(X, \mathcal{P}_1)$, $(X, \mathcal{P}_2)$ and $(X, \mathcal{P}_3)$ are three topological spaces and $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is a tritoplogical space.

Definition 3.2.3: A subset A of a tritopological space $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is said to be

(i) (i,j,k)-regular open (Banerjee 1987) if A = iInt(jCl(kInt(A))), where $i \neq j \neq k$, i,j,k=1,2,3.

(ii) (i,j,k)-regular closed (Bose and Sinha 1981) if A = iCl(jInt(kCl(A))), where $i \neq j \neq k$, i,j,k=1,2,3.

(iii) (i,j,k)-semi-open (Bose 1981) if $A \subset$ iCl(jInt(kCl(A))), where $i \neq j \neq k$, i,j,k=1,2,3.

(iv) (i,j,k)-preopen (Jelic 1990) if $A \subset$ iInt(jCl(kInt(A))), where $i \neq j \neq k$, i,j,k=1,2,3.

Definition 3.2.4: A subset A of a tritopological space $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is said to be

(i,j,k)-semi-preopen (Khedr and Noiri 1992) if $\exists$ a (i,j,k)-preopen set U such that $U \subseteq A \subseteq$ jCl(kInt(U)) or it is said to be (i,j,k)-β-open if $A \subseteq$ jCl(kInt(iCl(A))), where $i \neq j \neq k$, i,j,k=1,2,3.

The complement of (i,j,k)-semi-preopen set is said to be (i,j,k)-semi-preclosed (Khedr and Noiri 1992) or is said to be (i,j,k)-β-closed if iInt(jCl(kInt(A))) $\subseteq$ A, where $i \neq j \neq k$, i,j,k=1,2,3.

Lemma 3.2.5: Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space and $\{A_\lambda : \lambda \in \Delta\}$ be a family of subsets of X. Then

(1) if A_λ is (i,j,k)-β-open for each $\lambda \in \Delta$, then $\bigcup_{\lambda\in\Delta} A_\lambda$ is (i,j,k)-β-open

(2) if A_λ is (i,j,k)-β-closed for each $\lambda \in \Delta$, then $\bigcap_{\lambda\in\Delta} A_\lambda$ is (i,j,k)-β-closed.

Proof: (1) The proof follows from Theorem 3.2 of (Khedr and Noiri 1992).

(2) This is an immediate consequence of (1) .

Definition 3.2.6: Let A be a subset of a tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$)

(1) The (i,j,k)-β-closure (Khedr and Noiri 1992) of A, denoted by (i,j,k)-βCl(A) is defined to be the intersection of all (i,j,k)-β-closed sets containing A.

(2) The (i,j,k)-β-interior of A, denoted by (i,j,k)-βInt(A) is defined to be the union of all (i,j,k)-β-open sets contained in A.

Lemma 3.2.7: Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space and A be a subset of X. Then

(1) (i,j,k)-βInt(A) is (i,j,k)-β-open

(2) (i,j,k)-βCl(A) is (i,j,k)-β-closed

(3) A is (i,j,k)-β-open iff A = (i,j,k)-βInt(A)

(4) A is (i,j,k)-β-closed iff A = (i,j,k)-βCl(A).

Proof: (1) and (2) are obvious from Lemma 3.2.3, (3) and (4) are obvious from (1) and (2).

Lemma 3.2.8: For any subset A of a tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$), $x \in$ (i,j,k)-βCl(A) iff $U \cap A \neq \phi$ for every (i,j,k)-β-open set U containing x.

Proof: The proof is trivial.

Lemma 3.2.9: Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space and A be a subset of X. Then

(1) X-(i,j,k)-βInt(A) = (i,j,k)-βCl(X-A)

(2) X-(i,j,k)-βCl(A) = (i,j,k)-βInt(X-A).

Proof: (1) By Lemma 3.2.5, (i,j,k)-βCl(A) is (i,j,k)-β-closed. Then X-(i,j,k)-βCl(A) is (i,j,k)-β-open. On the other hand, X - (i,j,k)-βCl(X-A)$\subseteq$A and hence X - (i,j,k)-βCl(X-A)$\subseteq$(i,j,k)-βInt(A). Conversely, let $x \in$(i,j,k)-βInt(A) . Then $\exists$ (i,j,k)-β-open set G such that $x \in G \subseteq A$. Then X-G is (i,j,k)-β-closed and X-A$\subseteq$X-G. Since $x \notin$X-G, $x \notin$(i,j,k)-βCl(X-A) and hence (i,j,k)-βInt(A)$\subseteq$X-(i,j,k)-βCl(X-A).

Hence X-(i,j,k)-βInt(A) = (i,j,k)-βCl(X-A).

(2) This follows immediately from (1).

Definition 3.2.10: Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space and A be a subset of X. A point x of X is said to be in the (i,j,k)-θ-closure (Kariofillis 1986) of A, denoted by (i,j,k)-cl_θ(A) if for every i-open set U containing x, A$\cap$jCl(kInt(U))$\neq \phi$, where i$\neq$j$\neq$k, i,j,k=1,2,3.

A subset A of X is said to be (i,j,k)-θ-closed if A=(i,j,k)-cl_θ(A). A subset A of X is said to be (i,j,k)-θ-open if X-A is (i,j,k)-θ-closed.

The (i,j,k)-θ-interior of A, denoted by (i,j,k)-Int_θ(A) is defined as the union of all (i,j,k)-θ-open sets contained in A . Hence $x \in$(i,j,k)-Int_θ(A) iff $\exists$ a i-open set U containing x such that $x \in$U$\subseteq$jCl(kInt(U))$\subseteq$A.

Lemma 3.2.11: For any subset A of a tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) the following properties hold:

(1) X-(i,j,k)-Int_θ(A) = (i,j,k)-cl_θ(X-A)

(2) X-(i,j,k)-cl_θ(A) = (i,j,k)-Int_θ(X-A) .

Lemma 3.2.12: (Kariofillis 1986) Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space. If U is a k-open set of X, then (i,j,k)-cl_θ(U) = iCl(jInt(U)).

Definition 3.2.13: A mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is said to be (i,j,k)-β-continuous (Khedr and Noiri 1992) if $f^{-1}(V)$ is (i,j,k)-β-open in X for each $\mathcal{Q}_i$ -open set V of Y.

Example 3.2.14: Consider the following tritopologies on X $=\{a,b,c\}$ and Y$=\{p,q,r\}$ respectively:

$\mathcal{P}_1 = \{X,\phi,\{a\},\{a,b\}\}$; $\mathcal{P}_2 = \{X,\phi,\{a\},\{b,c\}\}$; $\mathcal{P}_3 = \{X,\phi,\{a\},\{b,c\}\}$

and $\mathcal{Q}_1=\{Y,\phi,\{p\},\{p,r\}\}$; $\mathcal{Q}_2=\{Y,\phi,\{p\}\}$; $\mathcal{Q}_3=\{Y,\phi,\{q\}\}$

We define the mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) by f(a) = p, f(b) = q and f(c) = r. Then f is (1,2,3)- β-continuous since the inverse of each member of the topology $\mathcal{Q}_i$ - on Y is a (1,2,3)- β-open set in (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) .

Definition 3.2.15: (1) A mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is said to be (i,j,k)- weakly precontinuous if for each $x \in X$ and each $\mathcal{Q}_i$ -open set V of Y containing f(x), $\exists$ (i,j,k)- preopen set U containing x such that f(U)$\subseteq$jCl(kInt(V)).

Example 3.2.16: Consider the following tritopologies on X $=\{a,b,c\}$ and Y $=\{p,q,r\}$ respectively:

$\mathcal{P}_1 = \{X,\phi,\{a\},\{a,b\}\}$; $\mathcal{P}_2 = \{X,\phi,\{a\},\{b,c\}\}$; $\mathcal{P}_3 = \{X,\phi,\{a\},\{b\},\{a,b\}\}$

and $\mathcal{Q}_1 =\{Y,\phi,\{p\},\{p,r\}\}$; $\mathcal{Q}_2 =\{Y,\phi,\{p\}\}$; $\mathcal{Q}_3 =\{Y,\phi,\{q\}\}$

We define the mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) by f(a) = p, f(b) = q and f(c) = r. Then f is (1,2,3)- weakly precontinuous. Since if $a \in$X and $\mathcal{Q}_1$- open set V= $\{p, r\}$, then we have (1,2,3)- preopen set U = $\{a\}$ such that f(U) $\subseteq$ $\mathcal{Q}_2$-Cl($\mathcal{Q}_3$-Int(V)).

(2) A mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is said to be (i,j,k) weakly- β- continuous if for each $x \in$X and each $\mathcal{Q}_i$ -open set V of Y containing f(x), $\exists$ (i,j,k)- β-open set U containing x such that f(U) $\subseteq$ jCl(kInt(V)).

Remark 3.2.17: Since every (i,j,k)-preopen set is (i,j,k)- β-open (Remark 3.1 of Khedr and Noiri 1992) ,every (i,j,k)- weakly precontinuous function is (i,j,k)- weakly- β- continuous for $i \neq j \neq k$, i,j,k=1,2,3. The converse is not true.

3.3. Characterization.

Theorem 3.3.1: For a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) the following properties are equivalent:

(1) f is (i,j,k)- weakly- β-continuous

(2) (i,j,k)- β-Cl(f^{-1}(kInt(jCl(iInt(B))))) $\subseteq$ f^{-1}(jCl(iInt(B))) for every subset B of Y

(3) (i,j,k)-β-Cl(f^{-1}(kInt(F))) $\subseteq f^{-1}$(F) for every (i,j,k)- regular closed set F of Y

(4) (i,j,k)-β-Cl(f^{-1}(Cl(V))) $\subseteq f^{-1}$(iCl(jInt(B))) for every $\mathcal{Q}_k$-open set V of Y

(5) f^{-1}(V) $\subseteq$ (i,j,k)-β-Int(f^{-1}(jCl(kInt(V)))) for every $\mathcal{Q}_i$-open set V of Y

Proof: (1)$\Rightarrow$(2) . Let B be any subset of Y. Suppose that $x \in$X- f^{-1}(jCl(iInt(B))) . Then $f(x) \in$ Y- jCl(iInt(B)) so that $\exists$ a $\mathcal{Q}_j$- open set V of Y containing f(x) such that V$\cap$B = ϕ , so V$\cap$kInt(jCl(iInt(B))) = ϕ and hence kCl(iInt(V))$\cap$kInt(jCl(iInt(B))) = ϕ. Therefore $\exists$ (i,j,k)- β- open set U containing x such that f(U)$\subseteq$kCl(iInt(V)). Hence we have U $\cap f^{-1}$(kInt(jCl(iInt(B)))) = ϕ and $x \in$X- (i,j,k)-β-Cl(f^{-1}(kInt(jCl(iInt(B))))) by Lemma 3.2.6 . Thus we have (i,j,k)-β-Cl(f^{-1}(kInt(jCl(iInt(B))))) $\subseteq f^{-1}$(jCl(iInt(B))) .

(2)$\Rightarrow$(3). Let F be any (i,j,k)-regular closed set of Y . Then F = iCl(jInt(kCl(F))) and we have (i,j,k)-β-Cl(f^{-1}(kInt(F))) = (i,j,k)-β-Cl(f^{-1}(kInt(iCl(jInt(kCl(F)))))) $\subseteq$ f^{-1} (iCl(jInt(kCl(F)))) = f^{-1}(F).

(3)$\Rightarrow$(4). For any $\mathcal{Q}_k$- open set V of Y , iCl(jInt(V)) is (i,j,k)-regular closed . Then we have (i,j,k)-β-Cl(f^{-1}(Cl(V))) $\subseteq$(i,j,k)-β-Cl(f^{-1}(kInt(iCl(V)))) $\subseteq$ f^{-1}(iCl(jInt(V))) .

(4)$\Rightarrow$(5). Let V be $\mathcal{Q}_i$- open set of Y. Then Y - jCl(kInt(V)) is σ_k- open set in Y and we have (i,j,k)-β-Cl(f^{-1}(Y-jCl(kInt(V)))) $\subseteq f^{-1}$(jCl(Y-jCl(kInt(V))) and hence X- (i,j,k)-β-Int(f^{-1}(jCl(kInt(V))))$\subseteq$ X-f^{-1}(jInt(kCl(V)))$\subseteq$X- f^{-1}(V). Thus we obtain f^{-1}(V) $\subseteq$(i,j,k)-β-Int(f^{-1}(jCl(kInt(V)))) .

(5)$\Rightarrow$(1). Let $x \in$X and V be a $\mathcal{Q}_i$- open set containing f(x) . We have $x \in f^{-1}$(V) $\subseteq$ (i,j,k)-β-Int(f^{-1}(jCl(kInt(V)))) . Put U = (i,j,k)-β-Int(f^{-1}(jCl(kInt(V)))) . By Lemma 3.2.5 , U is (i,j,k)- β- open set containing x and f(U)$\subseteq$jCl(kInt(V)). This shows that f is (i,j,k)- weakly- β-continuous .

Theorem 3.3.2: For a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) the following properties are equivalent:

(1) f is (i,j,k)- weakly- β-continuous

(2) f((i,j,k)- βCl(A)) $\subseteq$ (i,j,k)- $Cl_\theta(f(A))$ for every subset A of X

(3) (i,j,k)- βCl(f^{-1}(B)) $\subseteq f^{-1}$((i,j,k)- $Cl_\theta(f(B))$) for every subset B of Y

(4) (i,j,k)- βCl(f^{-1}(jInt((i,j,k)- $Cl_\theta(f(B))$))) $\subseteq f^{-1}$((i,j,k)- $Cl_\theta(f(B))$) for every subset B of Y.

Proof: (1)$\Rightarrow$(2) . Suppose that f is (i,j,k)- weakly- β-continuous . Let A be any subset of X, $x\in$(i,j,k)-βCl(A) and V be a $\mathfrak{T}_i$- open set of Y containing f(x) . Then $\exists$ (i,j,k)- β- open set U containing x such that f(U)$\subseteq$jCl(kInt(V)). Since $x\in$(i,j,k)-βCl(A) , by Lemma 3.2.6 , we obtain U$\cap$A $\neq\phi$ and hence $\phi\neq$ f(U) $\cap$ f(A)$\subseteq$jCl(kInt(V))$\cap$f(A) . Therefore , we obtain f(x)$\in$(i,j,k)- $Cl_\theta(f(A))$.

(2)$\Rightarrow$(3) . Let B be any subset of Y. Then we have f((i,j,k)-βCl(f^{-1}(B))) $\subseteq$ (i,j,k)- $Cl_\theta\left(f^{-1}(B)\right)\subseteq$(i,j,k)-$Cl_\theta(B)$ and hence (i,j,k)- βCl(f^{-1}(B)) $\subseteq f^{-1}$((i,j,k)- $Cl_\theta(f(B))$).

(3)$\Rightarrow$(4) . Let B be any subset of Y. Since (i,j,k)-$Cl_\theta(B)$ is $\mathfrak{T}_i$- closed set in Y , by Lemma 3.2.10 , (i,j,k)- βCl(f^{-1}(jInt((i,j,k)- $Cl_\theta(B)$))) $\subseteq f^{-1}$((i,j,k)- Cl_θ (jInt((i,j,k)- $Cl_\theta(B)$))) = f^{-1}(iCl(jInt((i,j,k)- Cl_θ(B)))) $\subseteq f^{-1}$(iCl(i,j,k)- Cl_θ(B)) = f^{-1}((i,j,k)- Cl_θ(B)) .

(4)$\Rightarrow$ (1) . Let V be any $\mathfrak{T}_k$- open set of Y . Then by Lemma 3.2.10 , V $\subseteq$jInt(iCl(V)) = jInt((i,j,k)-$Cl_\theta(V)$) and we have (i,j,k)- βCl(f^{-1}(V)) $\subseteq$(i,j,k)- βCl(f^{-1}

(jInt((i,j,k)- Cl_{θ}(B))) $\subseteq f^{-1}$((i,j,k)- Cl_{θ}(V)) = f^{-1}(iCl(jInt(V))). Thus we have (i,j,k)- βCl(f^{-1}(V)) $\subseteq f^{-1}$(iCl(jInt(V))). It follows from Theorem 3.3.1 that f is (i,j,k)- weakly- β-continuous .

Definition 3.3.3: A tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be (i,j,k)-regular (Kelly 1963) if for each $x \in$X and each $\mathcal{P}_i$- open set U containing x , $\exists$ a $\mathcal{P}_i$- open set V such that $x \in$V $\subseteq$jCl(kInt(V)) $\subseteq$U .

Example 3.3.4: Consider the following tritopologies on X = $\{a, b, c, d\}$:

$\mathcal{P}_1 = \{X, \phi, \{a\}, \{a,b,c\}, \{a,c\}, \{a,d\}\}$; $\mathcal{P}_2 = \{X, \phi, \{a, b\}, \{b,c\}, \{b\}\}$ and

$\mathcal{P}_3 = \{X, \phi, \{a, c\}, \{b,c\}, \{c\}\}$

Then X is (1,2,3) regular since for $a \in$ X , $\mathcal{P}_1$- open set U = $\{a,b,c\}$, $\exists$ $\mathcal{P}_1$- open set V = $\{a,c\}$ such that V $\subseteq \mathcal{P}_2$ –Cl($\mathcal{P}_3$- Int(V)) $\subseteq$U .

Lemma 3.3.5 : (Popa and Noiri 2004) If a tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is (i,j,k)-regular , then (i,j,k)- Cl_{θ}(F) = F for every $\mathcal{P}_i$- closed set F .

Theorem 3.3.6: Let (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) be an (i,j,k)- regular tritopological space . For a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) the following properties are equivalent:

(1) f is (i,j,k)- β-continuous

(2) f^{-1}((i,j,k)- Cl_θ(B)) is (i,j,k)- β- closed in X for every subset B of Y

(3) f is (i,j,k)- weakly - β-continuous

(4) f^{-1}(F) is (i,j,k)- β- closed in X for every (i,j,k)- θ- closed set F of Y

(5) f^{-1}(V) is (i,j,k)- β- open in X for every (i,j,k)- θ- open set V of Y .

Proof : (1)$\Rightarrow$ (2) . Let B be any subset of Y. Since (i,j,k)- $Cl_\theta(B)$ is $\mathcal{Q}_i$- closed set in Y , it follows by Theorem 5.1 of (Khedr and Noiri 1992) that f^{-1}((i,j,k)- Cl_θ(B)) is (i,j,k)- β- closed in X .

(2)$\Rightarrow$ (3) . Let B be any subset of Y. Then we have (i,j,k)- βCl(f^{-1}(B))) $\subseteq$ (i,j,k)- βCl(f^{-1}(i,j,k)- Cl_θ(B)) = f^{-1} (i,j,k)- $Cl_\theta(B)$. By Theorem 3.3.2 , f is (i,j,k)- weakly - β-continuous .

(3)$\Rightarrow$ (4). Let F be any (i,j,k)- θ- closed set of Y. Then by Theorem 3.3.2 , (i,j,k)- βCl(f^{-1}(F)) $\subseteq f^{-1}$ ((i,j,k)- Cl_θ(F)) = f^{-1}(F) . Therefore by Lemma 3.2.5 , f^{-1}(F) is (i,j,k)- β- closed in X .

(4)$\Rightarrow$ (5). Let V be any (i,j,k)- θ- open set of Y. By (4) , f^{-1}(Y – V) = X - f^{-1}(V) is (i,j,k)- β- closed in X and hence f^{-1}(V) is (i,j,k)- β- open in X .

(5)$\Rightarrow$ (1). Since Y is (i,j,k)- regular , by Lemma 3.3.5 , (i,j,k)- Cl_θ(B) = B for every $\mathcal{Q}_i$- closed set B of Y and hence $\mathcal{Q}_i$- open set is (i,j,k)- θ- open set . Therefore f^{-1}(V) is (i,j,k)- β- open for every $\mathcal{Q}_i$- open set V of Y . By Theorem 5.1 of (Khedr and Noiri 1992) , f is (i,j,k)- β-continuous .

3.4. Weakly -β-Continuity and β-Continuity.

Definition 3.4.1: A mapping f : (X , $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is said to be (i,j,k)- weakly* quasi continuous (briefly w*.q.c) (Popa and Noiri 2004) if for every $\mathcal{Q}_i$- open set V of Y , f^{-1}(jCl(kInt(V))) is triclosed in X .

Theorem 3.4.2: If a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is (i,j,k)- weakly-β-continuous and (i,j,k)- w*.q.c , then f is (i,j,k)- β-continuous .

Proof : Let $x \in$ X and V be any $\mathcal{Q}_i$- open set of Y containing f(x) . Since f is (i,j,k)-weakly-β-continuous , $\exists$ an (i,j,k)-β- open set U of X containing x such that f(U)$\subseteq$jCl(kInt(V)) .

Hence $x \notin f^{-1}$(jCl(kInt(V)) - V) . Therefore $x \in$U - f^{-1}(jCl(kInt(V)) - V) = U $\cap$ (X - f^{-1}(jCl(kInt(V)) - V)) . Since U is (i,j,k)-β - open and X - f^{-1}(jCl(kInt(V)) - V) is triopen , by Theorem 3.3 of (Khedr and Noiri 1992) , G = U $\cap$X - f^{-1}(jCl(kInt(V)) - V) is (i,j,k)-β - open . Then $x \in$G and f(G) $\subseteq$V . For if $y \in$G , then $f(y) \notin$(jCl(kInt(V)) - V) and hence f(y) $\in$ V . Therefore f is (i,j,k)- β-continuous .

Definition 3.4.3: A mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is said to have a (i,j,k)-β- interiority condition if (i,j,k)-βInt(f^{-1}(jCl(kInt(V))) $\subseteq f^{-1}$(V) for every $\mathcal{Q}_i$- open set V of Y .

Theorem 3.4.4: If a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is (i,j,k)-weakly-β-continuous and satisfies the (i,j,k)-β- interiority condition then f is (i,j,k)-β-continuous .

Proof : Let V be $\mathcal{Q}_i$- open set of Y . Since f is (i,j,k)- weakly-β-continuous , by Theorem 3.3.1 , f^{-1} (V) $\subseteq$(i,j,k)-βInt(f^{-1}(jCl(kInt(V))) . By (i,j,k)-β-

interiority condition of f , we have (i,j,k)-βInt(f^{-1}(jCl(kInt(V))) $\subseteq f^{-1}$(V) and hence (i,j,k)-βInt(f^{-1}(jCl(kInt(V))) = f^{-1}(V) . By Lemma 3.2.5 , f^{-1}(V) is (i,j,k)-β- open in X and thus f is (i,j,k)-β-continuous .

Definition 3.4.5: Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space and let A be a subset of X . The (i,j,k)-β- frontier of A is defined as (i,j,k)-βFr(A) = (i,j,k)-βCl(A)$\cap$(i,j,k)-βCl(X -A) = (i,j,k)-βCl(A) - (i,j,k)-βInt(A) .

Theorem 3.4.6: The set of all points x of X for which a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is not (i,j,k)- weakly-β-continuous is identical with the union of the (i,j,k)-β- frontier of the inverse images of the jCl(kInt(V)) of $\mathcal{Q}_i$- open set V of Y containing f(x).

Proof : Let x be a point of X at which f(x) is not (i,j,k)- weakly-β-continuous . Then $\exists$ a $\mathcal{Q}_i$- open set V of Y containing f(x) such that U $\cap$ (X - f^{-1}(jCl(kInt(V)))) $\neq \phi$ for every (i,j,k)-β- open set U of X containing x . By Lemma 3.2.6 , $x \in$(i,j,k)-βCl(X - f^{-1}(jCl(kInt(V)))) . Since $x \in f^{-1}$(jCl(kInt(V))) , we have $x \in$(i,j,k)-βCl(f^{-1} (jCl(kInt(V)))) and hence $x \in$(i,j,k)-βFr(f^{-1}(jCl(kInt(V)))) .

Conversely , if f is (i,j,k)- weakly-β-continuous at x , then for each $\mathcal{Q}_i$- open set V of Y containing f(x) , $\exists$ (i,j,k)-β- open set U containing x such that f(U) $\subseteq$jCl(kInt(V)) and hence $x \in U \subseteq f^{-1}$(jCl(kInt(V))) . Therefore we obtain that $x \in$(i,j,k)-βInt (f^{-1}(jCl(kInt(V)))) . This contradicts that $x \in$(i,j,k)-βFr(f^{-1}(jCl(kInt(V)))) .

3.5 Weakly - β-Continuity and Almost β-Continuity .

Definition 3.5.1: A mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is said to be (i,j,k)- almost β- continuous if for each $x \in$X and each $\mathcal{Q}_i$- open set V containing f(x), $\exists$ an (i,j,k)-β- open set U of X containing x such that f(U) $\subseteq$ iInt(jCl(kInt(V))) .

Lemma 3.5.2 : A mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is (i,j,k)- almost β- continuous iff f^{-1}(V) is (i,j,k)-β- open for each (i,j,k)-regular open set V of Y .

Definition 3.5.3: A tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be (i,j,k)- almost regular (Singal and Arya, 1971) if for each $x \in$X and each (i,j,k)-regular open set U containing x , $\exists$ an (i,j,k)-regular open set V of X such that $x \in V \subseteq$ jCl(kInt(V)) $\subseteq$ U .

Theorem 3.5.4: Let a tritopological space (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) be (i,j,k)- almost regular . Then a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is (i,j,k)- almost β- continuous iff it is (i,j,k)- weakly-β-continuous .

Proof : Necessity . This is obvious .

Sufficiency. Let us suppose that f is (i,j,k)- weakly-β-continuous . Let V be any (i,j,k)-regular open set of Y and $x \in f^{-1}$(V) . Then we have f(x)$\in$V . By the almost (i,j,k)-regularity of Y , $\exists$ an (i,j,k)-regular open set V_0 of Y such that $f(x) \in V_0 \subseteq$ jCl(kInt(V_0)) $\subseteq$ V . Since f is (i,j,k)- weakly-β-continuous , $\exists$ an (i,j,k)-β- open set U of X containing x such that f(U)$\subseteq$jCl(kInt(V_0))$\subseteq$V . This follows that $x \in$ U $\subseteq$ f^{-1}(V) . Therefore we have f^{-1}(V)$\subseteq$(i,j,k)-βInt(f^{-1}(V) . By Lemma 3.2.5 , f^{-1}(V) is (i,j,k)-β- open and by Lemma 3.5.2 , f is (i,j,k)- almost β- continuous .

Definition 3.5.5: A tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be triowise β-Hausdorff or triowise β-T_2 if for each distinct points x,y,z of X , $\exists$ (i,j,k) β- open set U, V and W containing x, y and z respectively such that U$\cap$V$\cap$W=ϕ for i$\neq$j$\neq$k, i,j,k=1,2,3.

Theorem 3.5.6: Let (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) be a tritopological space . If for each distinct points x, y, z in X , $\exists$ mapping f of (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) into triowise Hausdorff tritopological space (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) such that

(1) f(x) $\neq$ f(y) $\neq$ f(z)

(2) f is (i,j,k)- weakly-β-continuous at x

(3) f is (j,k,i)-almost-β-continuous at y

(4) f is (k,i,j)- almost-β-continuous at z .

Then (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is triowise β-Hausdorff .

Proof : Let x, y, z be three distinct points in X . Since Y is triowise Hausdorff , $\exists$ a $\mathcal{Q}_{i}$- open set U containing f(x) and a $\mathcal{Q}_{j}$- open set V containing f(y) and a $\mathcal{Q}_{k}$- open set W containing f(z) such that U$\cap$V$\cap$W = ϕ. Since U, V and W are disjoint we have kCl(U) $\cap$ kInt(jCl(V)) $\cap$ kCl(jInt(iCl(W)) = ϕ. Since f is (i,j,k)- weakly-β-continuous at x , $\exists$ an (i,j,k) β- open set U_x of X containing x such that f(U_x)$\subseteq$kCl(U) . Since f is (j,k,i)- almost-β-continuous at y , $\exists$ an (j,k,i) β- open set U_y of X containing y such that f(U_y)$\subseteq$kInt(jCl(V)) and since f is (k,i,j)-β- open set U_z of X containing z such that f(U_z)$\subseteq$kCl(jInt(iCl(W))) . Hence we have $U_x \cap U_y \cap U_z = \phi$. This shows that (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is triowise β-Hausdorff .

3.6 Some Properties

Definition 3.6.1: A tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be triowise Urysoshn (Bose and Sinha 1992) if for each distinct points x,y,z , $\exists$ i - open set U , j-open set V and k- open set W such that $x \in$U , $y \in$V and $z \in$W and jCl(U) $\cap$ kCl(V) $\cap$ iCl(W) =ϕ for i$\neq$j$\neq$k, i,j,k=1,2,3.

Theorem 3.6.2: If (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is triowise Urysohn and f : (X ,$\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is triowise weakly β- continuous injection , then (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is triowise $\beta - T_2$.

Proof : Let x,y,z be three distinct points of X . Then since f is injection, f(x)$\neq$ f(y)$\neq$f(z) . Since Y is triowise Urysohn , $\exists$ $\mathcal{P}_i$- open set U , $\mathcal{P}_j$- open set V and $\mathcal{P}_k$- open set W such that f(x) $\in$U, f(y) $\in$ V and f(z) $\in$ W and jCl(U)$\cap$ kCl(V) $\cap$ iCl(W)=ϕ for i$\neq$j$\neq$k, i,j,k=1,2,3. Hence f^{-1}(jCl(U)) $\cap$ f^{-1}(kCl(V)) $\cap$ f^{-1}(iCl(W)) = ϕ . Therefore (i,j,k)-βInt(f^{-1}(jCl(U))) $\cap$ (j,k,i)-βInt(f^{-1}(kCl(V))) $\cap$ (k,i,j)-βInt(f^{-1}(iCl(W))) = ϕ . Since f is triowise weakly β- continuous by Theorem 3.3.1, $x \in f^{-1}$(U) $\subseteq$(i,j,k)-βInt(f^{-1}(jCl(U))) , $y \in f^{-1}$(V) $\subseteq$(j,k,i)-βInt(f^{-1}(kCl(V))) , $z \in f^{-1}$(W) $\subseteq$(k,i,j)-βInt(f^{-1}(iCl(W))) . This implies that (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is triowise $\beta - T_2$.

Definition 3.6.3: A tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be triowise connected (Previn 1967) (resp. triowise β-connected) if it can not be expressed as the union of three non-empty disjoint sets U,V and W such that U is i-open , V is j-open and W is k-open (resp.(i,j,k)-β-open , (j,k,i)-β-open and (k,i,j)-β-open).

Theorem 3.6.4: If a mapping f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) → (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is triowise weakly β- continuous surjection and (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is triowise β-connected , then (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is triowise connected .

Proof: Suppose that (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is not triowise connected . Then ∃ a $\mathcal{Q}_i$- open set U , $\mathcal{Q}_j$- open set V and $\mathcal{Q}_k$- open set W such that U≠ϕ ,V ≠ϕ, W ≠ϕ , U∩V∩W=ϕ and U∪V∪W= Y . Since f is surjective , f^{-1}(U) , f^{-1}(V) and f^{-1}(W) are non-empty . Moreover , f^{-1}(U) ∩ f^{-1}(V) ∩ f^{-1}(W) = ϕ and f^{-1}(U) ∪ f^{-1}(V) ∪ f^{-1}(W) = X . Since f is triowise weakly β- continuous , by Theorem 3.3.1 , we have f^{-1}(U) ⊆ (i,j,k)-βInt(f^{-1}(jCl(U))) , f^{-1}(V) ⊆ (j,k,i)-βInt(f^{-1}(kCl(V))) and f^{-1}(W) ⊆ (k,i,j)-βInt(f^{-1}(iCl(W))) . Since U is j-closed and k-closed, V is i-closed and k-closed, W is i-closed and j-closed , we have f^{-1}(U) ⊆ (i,j,k)-βInt(f^{-1}(U)) , f^{-1}(V) ⊆ (j,k,i)-βInt(f^{-1}(V)) and f^{-1}(W) ⊆ (k,i,j)-βInt(f^{-1}(W)) . Hence f^{-1}(U) = (i,j,k)-βInt(f^{-1}(U)) , f^{-1}(V) = (j,k,i)-βInt(f^{-1}(V)) and f^{-1}(W) = (k,i,j)-βInt(f^{-1}(W)) . By Lemma 3.2.5 , f^{-1}(U) is (i,j,k)-β-open , f^{-1}(V) is (j,k,i)-β-open , f^{-1}(W) is (k,i,j)-β-open in (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) . This shows that (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is not triowise β-connected.

Definition 3.6.5: A subset A of a tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be (i,j,k)-quasi H-closed relative to X (Banerjee 1987) if for each cover $\{U_\alpha : \alpha \in J\}$ of A by $\mathcal{P}_i$-open sets of X , $\exists$ a finite subset J_0 of J such that A $\subseteq$ $\bigcup \{jCL(kInt(U_\alpha)) : \alpha \in J_0\}$.

Definition 3.6.6: A subset A of a tritopological space (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is said to be (i,j,k)-β-compact relative to X if every cover of A by (i,j,k)-β-open sets of X has a finite subcover .

Theorem 3.6.7: If f : (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) $\rightarrow$ (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) is triowise weakly β-continuous and A is (i,j,k)-β-compact relative to X , then f(A) is (i,j,k)-quasi H-closed relative to Y.

Proof : Let A be (i,j,k)-β-compact relative to X and $\{V_\alpha : \alpha \in J\}$ any cover of f(A) by $\mathcal{Q}_i$-open sets of (Y, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$) . Then f(A)$\subseteq \bigcup \{V_\alpha : \alpha \in J\}$ and so A$\subseteq \bigcup$ $\left\{f^{-1}(V_\alpha) : \alpha \in J\right\}$. Since f is (i,j,k)- weakly β- continuous , by Theorem 3.3.1, we have $f^{-1}(V_\alpha) \subseteq$ (i,j,k)-βInt(f^{-1}(jCl(kInt(V_α))) for each $\alpha \in J$. Therefore A $\subseteq \bigcup$ {(i,j,k)-βInt(f^{-1}(jCl(kInt(V_α)))} for each $\alpha \in J$. Since A is (i,j,k)-β-compact relative to X and (i,j,k)-βInt(f^{-1}(jCl(kInt(V_α))) is (i,j,k)-β-open for each

$\alpha \in J$, $\exists$ a finite subset J_0 of J such that A $\subseteq \bigcup$ {(i,j,k)- β Int(f^{-1}(jCl(kInt(V_α))) : $\alpha \in J_0$}. This implies that f(A)$\subseteq \bigcup$ {f((i,j,k)- β Int(f^{-1}(jCl(kInt(V_α)))) : $\alpha \in J_0$} $\subseteq \bigcup$ {f(f^{-1}(jCl(kInt(V_α))) : $\alpha \in J_0$ } $\subseteq \bigcup${jCl(kInt(V_α)) : $\alpha \in J_0$ } . Hence f(A) is (i,j,k)-quasi H-closed relative to Y.

CHAPTER FOUR

DENSITY TOPOLOGY IN TRITOPOLOGICAL SPACES

4.1. Introduction.

In this chapter we introduce the concept of density topology in a tritopological space and derive some relevant separation properties involving the density topology.

The idea of density topology has been widely studied in various spaces such as bitopological space, measure space, real number space, Romanvoski space etc. (see Goffman and Waterman (1961), Lahiri and Das (2002), Martin (1964), Saha and Lahiri (1989), A.K. Banerjee (2008).

We have generalized a work of Lahiri and Das (2002) to tritopological spaces.

In this chapter we attempt to define density of sets in a tritopological space $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ and with the idea of trioclosure we generate a topology which is helpful in study of some separation properties.

Here we study density of sets in a tritopologcial space satisfying certain axioms and investigate some relevant separation properties.

4.2. Tritopological Spaces.

Here we give some definitions in tritopological spaces.

Definition 4.2.1: A cover Ω of a tritoplogical space $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is said to be trio-pen if $\Omega \subset \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ and Ω contains at least one non-empty member from each of $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$. The space $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is said to be tricompact if every trio-open cover of it has a finite subcover.

Definition 4.2.2: Recall that if $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ be a tritopological space for any $A \subset X$, define $\overline{A} = \cap \{F_1 \cup F_2 \cup F_3$ where $A \subset F_1 \cup F_2 \cup F_3$ and F_1, F_2 and F_3 are respectively $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ - closed$\}$, then $\overline{A}$ is called the triowise closure of A.

When A is a subset of a tritopological space X (bitopological space X) by $\overline{A}$ we mean triowise closure of A (pairwise closure of A).

Theorem 4.2.3: Let X be a tritopological space and let $\mathcal{S} = \{V : V \subset X$ and $\overline{(X - V)} = X - V\}$, then $(X, \mathcal{S})$ is a topological space.

Note: We observe that if F is $\mathcal{P}_1$ or $\mathcal{P}_2$ or $\mathcal{P}_3$ closed then $\overline{F} \subset F \cup \varphi \cup \varphi = F$, so that $\overline{F} = F$. Hence F is $\mathcal{S}$ –closed. This implies that $\mathcal{S}$ is finer than $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$.

Lemma 4.2.4: The family $\{P \cap Q \cap R : P \in \mathcal{P}_1, Q \in \mathcal{P}_2 \text{ and } R \in \mathcal{P}_3\}$ forms a base for $\mathcal{S}$.

Proof: Clearly, the sets $P \cap Q \cap R$ belong to $\mathcal{S}$. If $V \in \mathcal{S}$ then

$X - V = \overline{(X - V)}$

$= \cap \{ F_1 \cup F_2 \cup F_3 : F_1 \cup F_2 \cup F_3 \supset X - V$ and F_1, F_2 and F_3 are respectively $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$- closed$\}$

Therefore $V = X - (X - V)$

$= \cup \{P \cap Q \cap R : P \cap Q \cap R \subset V, P \in \mathcal{P}_1, Q \in \mathcal{P}_2 \text{ and } R \in \mathcal{P}_3\}$

Hence the proof is complete.

4.3. Density of Sets in $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$.

Definition 4.3.1: $\mathcal{B}\subset\mathcal{P}_1\cup\mathcal{P}_2\cup\mathcal{P}_3$ is said to be triowise open base of $(X,\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ if $\mathcal{B}\cap\mathcal{P}_1$ form a base for $\mathcal{P}_1$, $\mathcal{B}\cap\mathcal{P}_2$ form a base for $\mathcal{P}_2$ and $\mathcal{B}\cap \mathcal{P}_3$ form a base for $\mathcal{P}_3$.

Definition 4.3.2: The σ-algebra generated by the class of all sets of the form $P \cup Q\cup R$, $P\in \mathcal{P}_1$, $Q\in \mathcal{P}_2$ and $R\in \mathcal{P}_3$ is called the class of triowise Borel sets.

Definition 4.3.3: (Noiri, Khedr and AL-Areefi,1992) A mapping $f : (X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) \rightarrow (Y, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3)$ is said to be triowise continuous if inverse image of every $\mathcal{Q}_1$-open (resp. $\mathcal{Q}_2$-open, $\mathcal{Q}_3$-open) set in Y is $\mathcal{P}_1$-open (resp. $\mathcal{P}_2$-open, $\mathcal{P}_3$ -open) in X.

In $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$, let ξ be the class of all triowise Borel sets. Let μ be a measure on ξ such that μ (X) is finite. We also assume μ to be non-zero for all non-void sets of the form $P\cap Q\cap R$, $P\in \mathcal{P}_1$, $Q\in \mathcal{P}_2$, $R\in \mathcal{P}_3$. Let μ* be the outer measure on P(X) generated by μ. Let ϑ be the class of all μ*-measurable sets and $\mathcal{A}$ be a class of sets from ξ.

Definition 4.3.4: By a decomposition (SOLOMON (1969)) ξ_V of $V\in\xi$ we mean a finite disjoint family

$\{A_1, A_2,............,A_n\} \subset \mathcal{A}$ such that

(i) $\bigcup_{i=1}^{n} Ai \subset V$ *and*

(ii) $\mu(V - \bigcup_{i=1}^{n} Ai) = 0$

The class $\mathcal{A}$ is called a triowise fundamental sets (SOLOMON (1969)) if the following axioms hold.

AXIOM I. $\mathcal{A}$ form a triowise open base of (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) (and hence also $\mathcal{A}\subset \mathcal{P}_1\cup \mathcal{P}_2\cup \mathcal{P}_3$).

AXIOM II. For any $A\in\mathcal{A}$ and $\in>0$ there is a decomposition ξ_A of A such that $A'\in\xi_A$ implies μ $(A')<\in$.

AXIOM III. For each triowise compact set W and for each $\mathcal{P}_1$ or, $\mathcal{P}_2$ or $\mathcal{P}_3$-open set $V\supset W$, there is an $\in>0$ such that if $A\in\mathcal{A}$ and μ $(A)<\in$ and $\overline{A}\cap W \neq \phi$ then $A\subset V$.

AXIOM IV: Given $A \in \mathcal{A}$ and $\in > 0$ there is a $A' \in \mathcal{A}$ such that $\overline{A} \subset A'$ and $\mu(A'-A) < \in$.

Let $x \in X$, since by Axiom I, $\mathcal{A}$ forms a triowise open base of $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ there exists a $A \in \mathcal{A}$ such that $x \in A$. Let $\in > 0$, by AXIOM II there exists a decomposition $\{A_1, A_2, \ldots A_n\}$ of A with $\mu(A_i) < \in$, i=1, 2, ..., n. We now prove that $x \in \overline{A_i}$ for some i. If not, let $x \notin \overline{A_i}$ for all i. Since $\overline{A_i}$ is the intersection of all $\mathcal{S}$-closed sets containing A_i, there exists a $\mathcal{S}$-closed set $\hat{A}_i$, containing A_i which does not contain x. Then clearly the set $G = A - \bigcup_{i=1}^{n} \hat{A}_i$ is $\mathcal{S}$-open and non-void ($x \in G$) and $G \subset A - \bigcup_{i=1}^{n} Ai$ and so by our assumption about μ, $0 < \mu(G) \leq \mu(A - \bigcup_{i=1}^{n} Ai)$ which contradicts the condition (ii) of definition 4.3.4.

Hence for $\in > 0$ there exists a A_i such that $x \in \overline{A_i}$ and $\mu(A_i) < \in$.

Consequently, for each $x \in X$ there exists a sequence of triowise fundamental sets $\{A_{n,x}\}$ such that $x \in \overline{A}_{n,x}$ and $\mu(A_{n,x}) < \frac{1}{n}$ $\forall$ n.

Definition 4.3.5: (Lahiri and Das (2002)) For $x \in X$ and $E \subset X$ the upper and lower outer density of E at x denoted respectively by $\overline{\varphi}^*(E, x)$, $\underline{\varphi^*}(E, x)$ are defined by

$$\overline{\varphi}^*(E,x) = \lim_{n \to \infty} \overline{\varphi^*_n}(E,x)$$

$$\underline{\varphi^*}(E,x) = \lim_{n \to \infty} \underline{\varphi^*_n}(E,x)$$

where,

$$\overline{\varphi_n^*}(E,x) = \sup \{m^*(E, A);\ x \in \overline{A},\ \mu(A) < \frac{1}{n},\ A \in \mathcal{A}\}$$

$$\underline{\varphi_n}^*(E,x) = \inf \{m^*(E, A);\ x \in \overline{A},\ \mu(A) < \frac{1}{n},\ A \in \mathcal{A}\}$$

and $m^*(E, A) = \dfrac{\mu^*(E \cap A)}{\mu(A)}$.

Clearly $0 \le \underline{\varphi^*}(E,x) \le \overline{\varphi}^*(E,x) \le 1$. If they are equal, we denote the common value by $\varphi^*(E, x)$ and say the outer density of E exists at x. If $E \in \vartheta$ we write $\overline{\varphi}^*(E,x) = \overline{\varphi}(E,x)$ and $\underline{\varphi}^*(E,x) = \underline{\varphi}(E,x)$. If they are equal we write $\overline{\varphi}(E,x) = \underline{\varphi}(E,x) = \varphi(E,x)$.

We say x, an outer density point or an outer dispersion point of E according as $\underline{\varphi}$*(E, x) = 1 or $\overline{\varphi}$*(E,x)=0.

Theorem 4.3.6: (Theorem 4.1 of A.K. Banerjee (2008)) If E, F∈ϑ, φ(E,x), φ (F,x) exist and if E⊂F, then φ(F-E,x) exists and φ(F-E,x) = φ(F,x)-φ(E,x).

The proof is similar to the proof of theorem 3 of Saha and Lahiri (1989).

Definition 4.3.7: Let $\mathcal{D}$= {V:V⊂X and φ*(X-V,x)=0, ∀x∈V}. As in Martin (1964) one can verify that $\mathcal{D}$ is a topology on X which is called the density topology (or, in short d-topology) on (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$).

The following two theorems is a generalizations of theorems 3 and 4 of (Lahiri and Das (2002)) .

Theorem 4.3.8: If V is $\mathcal{S}$-open, then ∀ x∈ V the outer density of V exist at x and φ*(V,x) =1.

Proof: Let x∈V, so by Lemma 4.2.6 there exists P∈ $\mathcal{P}_1$, Q∈ $\mathcal{P}_2$**,** R∈ $\mathcal{P}_3$ such that x∈ P∩Q∩R⊂V. Since {x} triowise compact and {x}⊂ P , by Axiom III, there is ∈>0 such that A∈$\mathcal{A}$ and {x} ∩$\overline{A}$ ≠ϕ and μ (A)<∈ ⇒ A⊂P. Choose $_{n0}$ ∈$\mathcal{N}$ such that 1/$_{n0}$ <∈. Then ∀ n>$_{n0}$, A∈$\mathcal{A}$ and x∈ $\overline{A}$ and μ (A) <1/n will imply A⊂ P.

Similarly, we can find $n_1 \in \mathcal{N}$ such that $\forall\ n \geq n_1$, $A \in \mathcal{A}$ and $x \in \overline{A}$ and $\mu(A) < 1/n \Rightarrow A \subset Q$ and $n_2 \in \mathcal{N}$ such that $\forall\ n > n_2$, $A \in \mathcal{A}$ and $x \in \overline{A}$ and $\mu(A) < 1/n \Rightarrow A \subset R$. Then $\forall\ n \geq m = \max\ \{n_0, n_1, n_2\}$, $A \in \mathcal{A}$ and $x \in \overline{A}$ and $\mu\ (A) < 1/n \Rightarrow A \subset P \cap Q \cap R \subset V$. Hence from definition of $\underline{\varphi_n}^*$ (V, x) it follows that $\forall\ n \geq m$.

$$\underline{\varphi_n}^*(V,x) = \inf \frac{\mu^*(V \cap A)}{\mu(A)}, A \in \mathsf{A},\ x \in \overline{A}, \mu(A) < 1/n$$

$$= 1, \text{ since } A \subset V.$$

Therefore $\underline{\varphi}^*(V,x)=1$ and hence $\varphi^*(V,x)=1$.

Theorem 4.3.9: The d- topology $\mathcal{D}$ is finer than $\mathcal{S}$.

Proof: Since $\mathcal{S} \subset \xi \subset \vartheta$ so by theorems 4.3.6 and 4.3.8 $V \in \mathcal{S}$ implies

$$\varphi\ (X-V,x) = \varphi(X,x) - \varphi\ (V,x)$$

$$= 1 - 1$$

$$= 0\ \forall\ x \in V.$$

Therefore $\mathcal{S} \subset \mathcal{D}$. This completes the proof.

4.4. Separation Properties in (X, $\mathcal{S}$).

Theorem 4.4.1: (Theorem 5 of Lahiri and Das (2002)) (X, $\mathcal{S}$) is regular.

Proof: Let E be $\mathcal{S}$-closed and $x \notin E$. Then $x \in X-E \in \mathcal{S}$ and so by Lemma 4.2.6, there are $P \in \mathcal{P}_1$, $Q \in \mathcal{P}_2$ and $R \in \mathcal{P}_3$ such that $x \in P \cap Q \cap R \subset X-E$. We associate with x, a sequence of triowise fundamental sets $\{A_{n,x}\}$ such that $x \in \overline{A}_{n,x}$ and $\mu(A_{n,x}) < 1/n \ \forall$ n. By Axiom IV for $A_{2n,x}$ there is $B_{n,x} \in \mathcal{A}$ and a $\mathcal{S}$-closed sets $\hat{A}_{2n,x}$ such that $x \in \overline{A}_{2n,x} \subseteq A_{2n,x} \subset B_{n,x}$ and $\mu(B_{n,x} - A_{2n,x}) < 1/2$ n. Then $\mu(B_{n,x}) \leq \mu(A_{2n,x}) + \mu(B_{n,x} - A_{2n,x}) < \frac{1}{2n} + \frac{1}{2n} = 1/n$. Thus we obtain a sequence $\{B_{n,x}\}$ from $\mathcal{A}$ such that $x \in B_{n,x}$ and $\mu(B_{n},x) < 1/n \ \forall n$. Again, proceeding as above for $B_{2n,x}$ we get a sequence $\{C_{n,x})$ from $\mathcal{A}$ satisfying $x \in B_{2n,x} \subset \overline{B}_{2n,x} \subset C_{n,x}$ and $\mu(C_{n,x}) < 1/n \ \forall$ n. Since $\{x\}$ is triowise compact and $x \in p$ so by Axiom III, there is $\in > 0$ such that $A \in \mathcal{A}$ and $\{x\} \cap \overline{A} \neq \varphi$ and $\mu(A) < \in$ implies $A \subset P$. Choose $n_o \in \aleph$ such that $1/n_\circ < \in$. Then $C_{no}, x \subset P$, since $x \in C_{n0,x} \subset \overline{C}_{n^\circ}, x$ and $\mu(C_{n0}, x) < 1/_{n0} < \in$. Now, $x \in B_{2n0}, x \subset \hat{B}_{2n0}, x \subset C_{n0}, x \subset P$.

Similarly, we can find $n_1 \in \aleph$ such that $x \in B_2n$, $x \subset B_{2n},x \subset C_{n1}$, $x \subset Q$ and for $n_2 \in \aleph$ such that $x \in B_2n_2$, $x \subset \hat{B}_{2n2}$, $x \subset C_{n2},x \subset R$.

Thus $x \in B_{2n0,x} \cap B_{2n1,x}, \cap B_{2n2,x} = U$ (say) $\subset \hat{B}_{2n0,x} \cap \hat{B}_{2n1,x} \cap \hat{B}_{2n2,x} = F$ (say) $\subset C_{no,x} \cap C_{n1,x} \cap C_{n2,x} \subset p \cap Q \cap R \subset X\text{-}E$, Hence we have $U \in \mathcal{S}$ (by AxiomI), $V = X\text{-}F \in \mathcal{S}$ satisfying $x \in U$, $E \subset V$ and $U \cap V = \phi$. Hence proved.

Corollary 4.4.2: (A.K. Banerjee (2008)) If $(X, \mathcal{S})$ is T_0 then it is T_2.

Proof: Let, x, y be two distinct points of X, then by T_0 there is a $\mathcal{S}$-open set U containing one of them say x, such that $y \in F = X\text{-}U$. Since F is $\mathcal{S}$-closed and $x \notin F$, so by regularity there are $\mathcal{S}$-open sets V,W such that $x \in V$, $F \subset W$ and $V \cap W = \phi$. Then $x \in V$, $y \in W$ and $V \cap W = \phi$. Therefore $(X, \mathcal{S})$ is T_2.

Definition 4.4.3: In $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$, $\mathcal{P}$ is said to be regular w.r.to $\mathcal{Q}$ if for any $\mathcal{P}_1$ -closed set F and $x \in X$ with $x \notin F$, $\exists\, U \in \mathcal{P}_1$, $V \in \mathcal{P}_2$ such that $x \in U$, $F \subset V$ and $U \cap V = \phi$. If $\mathcal{P}_1$ is regular w.r.to $\mathcal{P}_2$, and $\mathcal{P}_2$ is regular w.r.to $\mathcal{P}_3$ and $\mathcal{P}_3$ is regular w.r.to $\mathcal{P}_1$ then $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is called (1,2,3)-regular. If the space $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is (i,j,k)- regular then it is called triowise regular for $i \neq j \neq k$, i, j, k = 1,2,3.

If we consider the bitopological space the (X, $\mathcal{P}_1$, $\mathcal{P}_2$) and define $\mathcal{T}$ = {U: U⊂X and $\overline{X-U}$ = X-U} then (X,$\mathcal{T}$) is a regular toplogical space by Lahiri and Das (2002).

This topology $\mathcal{T}$ is finer than both $\mathcal{P}_1$ and $\mathcal{P}_2$. By Lemma 2 of Lahiri and Das (2002), we have {P ∩Q: P ∈ $\mathcal{P}_1$ and Q∈ $\mathcal{P}_2$} form a base for $\mathcal{T}$. Therefore by Lemma 4.2.6, $\mathcal{S}$ is finer than $\mathcal{T}$.

Then we have the following corollary.

Corollary 4.4.4: In (X, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{T}$) $\mathcal{S}$ is regular with respect to $\mathcal{D}$ and $\mathcal{T}$ is regular with respect to $\mathcal{S}$ and $\mathcal{D}$.

Note: When we say that $\mathcal{T}$ is regular with respect to $\mathcal{D}$, without loss of generality, we assume that $\mathcal{D}$ is the d- topology on (X, $\mathcal{P}_1$, $\mathcal{P}_2$) .

Theorem 4.4.5: (X, $\mathcal{D}$,$\mathcal{S}$,$\mathcal{T}$) is triowise regular if the following condition:

(a) For any $\mathcal{D}$-closed set E, if {A_n} is a sequence of triowise fundamental sets such that μ^* (E∩A_n)→0 as n→∝, there is at least one k∈ $\mathcal{N}$ such that E∩A_k=φ holds.

Proof: We only need to show that in X, $\mathcal{T}$ is regular with respect to $\mathcal{S}$, $\mathcal{S}$ is regular with respect to $\mathcal{D}$ and $\mathcal{D}$ is regular with respect to $\mathcal{T}$. By corollary 4.4.4 it is sufficient to show that $\mathcal{D}$ is regular with respect to $\mathcal{T}$. Let E be $\mathcal{D}$-closed and $x \notin E$. As in Theorem 4.4.1 we construct two sequences $\{B_{n,x}\}$, $\{C_{n,x}\}$ from $\mathcal{A}$ such that $x \in B_{2n,x} \subset \overline{B}_{2n,x} \subset C_{n,x}$ and $\mu(C_{n,x}) < 1/n\ \forall\ n$. Since X-E is $\mathcal{D}$-open and $x \in X-E$, $\overline{\varphi}^*(E,x)=0$. Let $\in > 0$ be arbitrary. Then there is $n_0 \in \mathcal{N}$ such that $1/n < \in$ and $\overline{\varphi_n}^*(E,x) < \in \forall\ n \geq n_0$. Since $x \in C_{n,x} \subset \overline{C}_{n,x}$, $\mu(C_{n,x}) < 1/n$, $m^*(E, C_{n,x}) < \in \forall n \geq n_0$ i.e, $\mu^*(E \cap C_{n,x})/\mu(C_{n,x}) < \in$ i.e; $\mu^*(E \cap C_{n,x}) < \in$. $\mu(C_{n,x}) < \in\ \forall\ n \geq n_0$. Thus $\mu^*(E \cap C_{n,x}) \rightarrow 0$ as $n \rightarrow \infty$. By the condition (a) there is $k \in \mathcal{N}$ such that $E \cap C_{k,x} = \phi$. Hence using Axiom I, $x \in B_{2k,x} \in \mathcal{T} \subset \mathcal{D}$, $E \subset X - C_{k,x} \subset X - \overline{B}_{2k,x} \in \mathcal{T}$ and $B_{2k,x} \cap (X - \overline{B}_{2k,x}) = \phi$. This proves the theorem.

Definition 4.4.6: $(X, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is said to be triowise Hausdorff if for every $x, y, z \in X$, $x \neq y \neq z\ \exists\ U \in \mathcal{P}_1$, $V \in \mathcal{P}_2$ and $W \in \mathcal{P}_3$ such that $x \in U$, $y \in V$, $z \in W$ and $U \cap V \cap W = \phi$.

Example 4.4.7: Consider the following tritopologies on $X = \{a, b, c\}$:

$$\mathcal{P}_1 = \{X, \phi, \{a\}, \{a,b\}\}\ ,\ \mathcal{P}_2 = \{X, \phi, \{b,c\}, \{b\}\}\ ,\ \mathcal{P}_3 = \{X, \phi, \{a,c\}, \{c\}\}$$

Then X is (1,2,3) Hausdorff since for $a,b,c \in X$, $\mathcal{P}_1$- open set $U = \{a\}$, $\mathcal{P}_2$- open set $V = \{b\}$, $\mathcal{P}_3$- open set $W = \{c\}$, then we have $U \cap V \cap W = \phi$.

Theorem 4.4.8: If (X, $\mathcal{P}_1$) or (X, $\mathcal{P}_2$) or (X, $\mathcal{P}_3$) is T_1, then (X, $\mathcal{D}$, $\mathcal{T}$) is pairwise Hausdorff. Also if (X, $\mathcal{D}$, $\mathcal{T}$) is pairwise Hausdorff then (X, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{T}$) is triowise Hausdorff.

Proof: The proof of the first part is similar to Theorem 7 of Lahiri and Das (2002).

For the last part of the theorem, let (X, $\mathcal{D}$, $\mathcal{T}$) be pairwise Hausdorff. Let x, y, z$\in$X with $x\neq y\neq z$. Since (X, $\mathcal{D}$, $\mathcal{T}$) is pairwise Hausdorff, for x, y $\exists$ $U'\in\mathcal{D}$, $V'\in\mathcal{T}$ such that $x\in U'$, $y\in V'$ and $U'\cap V'=\phi$, for y,z $\exists$ $U''\in\mathcal{D}$, $V''\in\mathcal{T}$ such that $y\in U''$, $z\in V''$ and $U''\cap V''=\phi$ and also for x, z $\exists$ $U'''\in\mathcal{D}$, $V'''\in\mathcal{T}$ such that $x\in U'''$, $z\in V'''$ and $U'''\cap V'''=\phi$. Put $U'\cap U'''=U$, $V'=V$ and $V''\cap V'''=W$. Then $x\in U\in\mathcal{D}$, $y\in V\in\mathcal{T}\subset\mathcal{S}$, $z\in W\subset\mathcal{T}$ and clearl,$U\cap V\cap W=\phi$.

Therefore (X, $\mathcal{D}$, $\mathcal{S}$, T) is triowise Hausdorff.

Definition 4.4.9: (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is called (1,2,3) normal if for any pairwise disjoint $\mathcal{P}_1$-closed set A, $\mathcal{P}_2$-closed set B, $\mathcal{P}_3$-closed set C, $\exists$ $U\in\mathcal{P}_1$, $V\in\mathcal{P}_2$, $W\in\mathcal{P}_3$ such that $A\subset V$, $B\subset W$, $C\subset U$ and $U\cap V\cap W=\phi$.

If (X, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$) is (i, j, k) normal then it is called triowise normal for $i\neq j\neq k$, i,j,k=1,2,3.

Theorem 4.4.10: If (X, $\mathcal{D}$) is compact then (X, $\mathcal{D}$, $\mathcal{T}$, $\mathcal{S}$) is triowise normal.

Proof: Let A, B and C be pairwise disjoint $\mathcal{D}$-closed, $\mathcal{T}$ -closed and $\mathcal{S}$-closed sets respectively. Since (X, $\mathcal{S}$) is regular for any $x\in A$, $\exists$ U_x, $V_x \in \mathcal{S}$ such that $x\in U_x, C\subset V_x$ and $U_x\cap V_x= \phi$. Now, $\{U_x: x\in A\}$ form a $\mathcal{S}$-open cover of A and hence $\mathcal{D}$-open cover of A.

Since A is $\mathcal{D}$-closed $\exists$ $x_1, x_2, \ldots, x_n \in A$ such that $A\subset \bigcup_{i=1}^{n} Ux_i = U' \in \mathcal{S}$, $C\subset \bigcap_{i=1}^{n} Vx_i =$ $W\in \mathcal{S}\subset \mathcal{D}$, $U'\cap V= \phi$. Also since (X, $\mathcal{T}$) is regular for each $x\in A$, $\exists$ $U'x$, $V'x \in \mathcal{T}$ such that $x\in U'x$, $B\subset V'x$ and $U'x\cap V'x= \phi$. Also $\{U'x: x\in A\}$ form a $\mathcal{T}$ open cover of A and hence $\mathcal{D}$-open cover of A. Since A is $\mathcal{D}$-closed $A\subset \bigcup_{i=1}^{n} U'x_i = U\in \mathcal{T}$, $B\subset \bigcap_{i=1}^{n} V'x_i =$ $V\in U'\in \mathcal{T}\subset\mathcal{S}$, $U\cap W= \phi$. Therefore , we have $A\subset U\in\mathcal{T}$, $B\subset V\in\mathcal{S}$, $C\subset W\in \mathcal{D}$ and $U\cap V\cap W = \phi$.

Thus (X, $\mathcal{D}$, $\mathcal{T}$, $\mathcal{S}$) is (1,2,3) normal. Similarly, we can show that (X, $\mathcal{D}$, $\mathcal{T}$, $\mathcal{S}$) is (2,1,3) normal. Therefore (X, $\mathcal{D}$, $\mathcal{T}$, $\mathcal{S}$) is triowise normal.

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