Dedicated to

My parents, my wife, my beloved son, my niece and Tuhin's parents.

Solitary Wave Solutions of NLEEs in Plasma Physics and Engineering

Dissertation Submitted to the Department of Applied Mathematics in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

By

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June 2015

Declaration

I do hereby declare that the Ph.D. dissertation entitled “Solitary Wave Solutions of NLEEs in Plasma Physics and Engineering” is my original research work. Appropriate credit is given when ideas from other people’s works are used. The dissertation has not been formerly submitted anywhere for any degree.

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Certificate

I do hereby certify that the dissertation entitled “Solitary Wave Solutions of NLEEs in Plasma Physics and Engineering” submitted by Mr. Md. Ashrafuzzaman Khan in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Applied Mathematics, under the Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh has been completed under my supervision. I believe that this dissertation is original one and it has not been submitted anywhere for any degree or diploma. I have gone through the final draft of the thesis and found it worth submission.

Therefore, I recommend and forward the thesis to the University of Rajshahi for necessary formalities towards the award of the degree of Doctor of Philosophy in Applied Mathematics.

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List of Publications

1. **M. Ashrafuzzaman Khan** and M. Ali Akbar, Exact and Solitary Wave Solutions to the Fifth-order KdV Equation by Using the Modified Simple Equation Method, **Applied and Computational Mathematics**, 4(3) (2015) 122-129. DOI: 10.11648/j.acm.20150403.14.
2. **M. Ashrafuzzaman Khan** and M. Ali Akbar, Solitary Wave Solutions to the Strain Wave Equation in Microstructured Solids through the Modified Simple Equation Method, **Physical Science International Journal**, (2015) (accepted for publication), **(proof has been attached in the next page).**
3. **M. Ashrafuzzaman Khan** and M. Ali Akbar, Solitary Wave Solutions to the Strain Wave equation in Microstructured Solids and Fifth-order KdV equation through the (G'/G)-expansion method, **Applied and Computational Mathematics**,, (submitted, 2015).

Abstract

Although the modified simple equation (MSE) method effectively provides exact solitary wave solutions to nonlinear evolution equations (NLEEs) in the field of applied mathematics, mathematical physics, plasma physics and engineering, it has some limitations. When the balance number is greater than one, usually the method does not give any solution. In this dissertation, we have exposed a process as to how to implement the MSE method to solve the NLEEs for balance number two. In order to verify the process, some NLEEs have been solved by means of this scheme, and we found some fresh traveling wave solutions. When the parameters receive special values, solitary wave solutions are derived from the exact traveling wave solutions and we have analyzed the solitary wave properties by the graphs of the solutions. These solitary wave solutions include soliton, kink shape soliton, singular kink shape soliton, bell shape soliton, singular bell shape soliton, anti-bell shape soliton, singular anti-bell shape soliton, etc. The attraction of the MSE method is that it is consistent, peaceful, authentic, and we found some fresh new traveling wave solutions other than the existing methods, such as, the basic (G'/G)-expansion method. We emphasize the implementation of the MSE method, how to examine the solutions to NLEEs for balance number two and also compare the solutions obtained by the MSE method and the well-known existing (G'/G)-expansion method. This shows the validity, usefulness, and necessity of the MSE method and our graphical representations describe the obtained traveling wave solutions.

Chapter 1

Introduction

The mathematical modeling of physical phenomena that change over time depends closely on the system of differential equations, namely, ordinary and partial differential equations. Through the diverse fields, like the natural and physical sciences, economics, epidemiology, neural networks, bioscience, mechanics etc., the mathematical models are developed to study these phenomena. In spite of the fact that these models provided the nature of the fields, the adequacy of their contribution to the common characteristics that make it possible to examine the different groups of them within a unified theoretical structure or mathematical form of the differential equations, which are linear, nonlinear, homogeneous or non-homogeneous equations. Solution procedures to linear differential equations are relatively easy and well recognized. But for the nonlinear equations it is not so easy to solve them and in some cases it is not possible and in general, approximations are typically used. Many scientists observed the fascinating element in the nature of nonlinear and for the fundamental understanding of nature, the science of nonlinear is the most important border. This study is an area of functional analysis, usually called the theory of evolution equations. These equations are basically nonlinear evolution equations (NLEEs). Therefore, in the natural sciences, it is very significant to study the solutions of NLEEs to uncover the obscurity of many events and processes.

The mathematical form of a NLEE is $u_t = f(u)$, *i. e.* a nonlinear partial differential equation with respect to time derivative. The linear heat equations or wave equations recounting heat conduction or vibration of cord are two simple examples of evolution equations. However, there are many nonlinear evolution equations which usually arising

from physics, plasma physics, mechanics, engineering, biology, chemistry, electrical circuits, solid state physics, high-energy physics, condensed matter physics, meteorology, oceanic phenomena, quantum mechanics, optical fibers, elastic media, fluid mechanics, acoustics, protein chemistry, mathematical biology, water surface gravity waves, ion acoustic waves in plasma, material science etc., should be investigated. Actually almost all the evolution equations concerning physical phenomena are nonlinear. The properties of each nonlinear equation are distinct and every nonlinear equation has its own peculiarity. Therefore, complication of NLEEs has drawn a lot of attention of many mathematicians and scientist who are involved with nonlinear science. For better understanding of the inner structure of the phenomena, as well as their further applications in practical life, exact solutions might play a fundamental role. The exact solutions can explain the problems precisely and the physical significance of the system duly. In addition to the physical significance, the close-form solutions to NLEEs assist the numerical solvers to evaluate the precision of their results and help them in the stability analysis. Exact solutions contain some arbitrary constants, when the constants receive some particular values solitary wave solutions are originated from the exact traveling wave solutions. A solitary wave is a localized gravity wave that occurs from the balance between dispersive effects and nonlinearity. When a solitary wave remains in its shape and velocity during its collision with another wave of the same kind, though possibly for a phase shift, is call a soliton. *i. e.* it can be treated like a particle that upholds its profile when it travels at constant speed. Equations with soliton solutions have a profound mathematical structure. Solitary waves arise equally in continuous and discrete systems and in both one and multiple spatial dimensions. Key issues in studying solitary waves also include linear versus nonlinear, persistent versus transient, integrable versus non-

integrable, asymptotics, localization in physical space versus Fourier space, and the effects of noise.

Therefore, it is significant to seek as many solitary wave solutions as possible to the NLEEs. In the recent years, significant efforts have been made by various groups of scientists to find solitary wave solutions to the NLEEs. All nonlinear equations cannot be solved by a particular method. Therefore, they established several methods to obtain exact solitary wave solutions, such as, the inverse scattering method, the Backlund transformation method, the Adomian decomposition method, the variational iteration method, the He's homotopy perturbation method, the Jacobi elliptic function method, the homogeneous balance method, the tanh-function method, the sine-cosine method, the F-expansion method, the Exp-function method, the ansatz method, the (G'/G)-expansion method, the modified simple equation (MSE) method, the $\exp(-\varphi(\eta))$-expansion method, etc.

The MSE method is an effective method in searching exact solitary wave solutions to NLEEs, but the method has some shortcomings. When the balance number is greater than one, typically there arise difficulties in solving the NLEEs by means of the MSE method. One cannot use the MSE method in straight away. In this dissertation, we have by using the MSE method established a procedure to examine the exact solitary wave solutions to NLEEs whose balance number is two. Inserting the assumed solution to the corresponding ordinary differential equation and then equating the coefficients of $(\psi(\xi))^{-j}$, $j = 0, 1, 2, 3, \cdots$ yield an over-determined set of algebraic and differential equations. During determination of the unknown function, there is born a third order linear ordinary differential equation in $\psi(\xi)$ and ξ. If in the solution of $\psi(\xi)$, ξ appears as a polynomial, it will not be eligible to be received as solitary wave solution, because for

solitary wave solution, $|u| \to 0$ as $\xi \to \infty$. Therefore, the coefficients of the polynomial must be zero. This constraint is essential to solve NLEEs for higher balance number. This procedure plays a very important role and can be applied to many NLEEs. The solutions obtained by this technique might play very important role in the field of applied mathematics, mathematical physics, plasma physics and engineering. We have analyzed and illustrated the solitary wave properties of the solutions by graph.

Chapter 2

The Literature Review and Proposal

Preview Material

- 2.1: The Literature Review
- 2.2: Wave and Soliton
- 2.3: Motivation
- 2.4: Objectives
- 2.5: The Proposal

2.1: The Literature Review

In general physical systems are explained by nonlinear partial differential equations. The mathematical modeling of intricate phenomena in applied mathematics, physics, mathematical physics, bioscience, medical science, plasma physics, microstructured solid materials in engineering fields that change over time are closely related to the study of variety of systems of ordinary and partial differential equations. Similar models have also evolved in various fields of study, ranging from the natural and physical sciences, population ecology to economics, infectious disease epidemiology, neural networks, biology, mechanics etc. Therefore, mathematical theories are very much important to find solutions to nonlinear partial differential equations because physical systems are generally explained by nonlinear equations. In spite of the eclectic nature of the fields wherein these models are formulated, different groups of scientists contribute adequate common attributes that make it possible to examine them within a unified theoretical structure. Such study is an area of functional analysis usually called the theory of evolution

equations. Therefore, the investigation of solutions to NLEEs plays a very important role to uncover the obscurity of many phenomena and processes throughout the natural sciences. But, one of the essential problems is to obtain their exact solutions. Because, nonlinear processes are one of the major confrontations and difficult to manage since the nonlinear characteristic of the system abruptly changes owing to slight changes of valid parameters along with time. Thus, the issue becomes further complex and hence needs ultimate solution. In this case, advance nonlinear techniques are important to solve the problems which are inherently nonlinear, especially those involving differential equations, dynamical systems and associated areas (Alam, 2014).

Therefore, in order to find out exact solutions to NLEEs, different groups of mathematicians, physicist, and engineers have been working tirelessly. With the development of symbolic computation software, like, Mathematica, Maple or Matlab, direct method for searching solutions to NLEEs have become an attractive area of research. This software assists us by computing the complicated and cumbersome algebraic and linear differential equations speedily and successfully. As a result, they have been able to contribute significantly. Accordingly, in the recent years, they have established several methods to search exact solutions to NLEEs, for instance, the Exp-function method (He and Wu, 2006; He et al., 2012; Akbar and Ali, 2012; Manafian and Zamanpour, 2013; Mohyud-Din et al., 2009a; Misirli and Gurefe, 2011; Naher et al., 2011a; Naher et al., 2012; Yildirim and Pinar, 2010; Zhang, 2010), the Darboux transformation method (Leble and Ustinov, 1993; Matveev and Salle, 1991; Rogers and Schief, 2002), the inverse scattering method (Ablowitz and Clarkson, 1991; Baldock et al., 1981; Ghosh and Nandy, 1999), the Hirota's bilinear method (Hirota, 1973; Hirota, 2004; Hirota and Satsuma, 1981), the Backlund transformation method (Jianming et al., 2011; Miura, 1978; Rogers and Shadwick, 1982), the symmetry method (Bluman and

Kumei, 1989; Olver, 1986), the homogeneous balance method (Wang, 1995; Zayed et al., 2004a), the tanh method (Abdou, 2007; El-Wakil and Abdou, 2007; Fan, 2000; Malfliet and Hereman, 1996; Nassar et al. 2011; Salas and Gomez, 2008; Sekulic et al., 2011; Yusufoglu and Bekir, 2008; Zayed et al., 2004b), the Painleve expansion method (Weiss et al., 1982), the Jacobi elliptic function method (Chen and Wang, 2005; Porubov, 1996; Xu, 2006), the unified algebraic method (Fan, 2002), the hyperbolic function method (Inc and Evans, 2004; Zayed et al., 2004b), the Adomian decomposition method (Helal and Mehana, 2006; Kaya, 2001; Kaya, 2004; McOwen, 2004), the generalized Riccati equation method (Yan and Zhang, 2001), the ansatz method (Hu , 2001; Hu and Zhang, 2001), the sine-cosine method (Wazwaz, 2004; Yusufoglu and Bekir, 2006; Yan and Zhang, 1999), the Miura transformation method (Bock and Kruskal, 1979), the first integral method (Taghizadeh and Mirzazadeh, 2011), the He's homotopy perturbation method (Ganji, 2006; Ganji and Rafei, 2006; Ganji et al., 2007), the Cole-Hopf transformation method (Salas and Gomez, 2010), the Lie group symmetry method (Guo and Lin, 2010), the auxiliary equation method (Sirendaoreji, 2007), the F-expansion method (Wang and Li, 2005; Wang and Zhou, 2003), the modified extended direct algebraic method (Soliman and Abdo, 2009), the parameter-expansion method (He and Shou, 2007), the variational iteration method (Mohyud-Din, 2008; Mohyud-Din et al., 2009b-c; Noor and Mohyud-Din, 2008; Noor et al., 2008), the $\exp(-\varphi(\eta))$-expansion method (Khan and Akbar, 2013a; Hafez et al., 2015; Rahman et al., 2014a-b; Roshid et al., 2014; Uddin et al., 2014), the Sumudu transform method (Belgacem, 2006; Belgacem, 2007; Belgacem, 2009; Belgacem, 2010; Belgacem and Karaballi, 2006; Belgacem et al., 2003; Chaurasia et al., 2012; Watugala, 1993), the multiple Exp-function algorithm (Ma and Zhu, 2012), the homotopy analysis method (Domairry et al., 2009; Joneidi et al., 2010), the generalized tanh-coth method (Gomez and Salas, 2008; Jawad, 2012), the

spectral-homotopy analysis method (Makukula et al., 2010; Motsa et al., 2010a-b; Sibanda et al., 2010) and so on.

Li et al. (Li et al., 2004) implemented the generalized Riccati equation expansion method, in which $G(\xi)$ satisfied an auxiliary equation: $G'(\xi) = C_1 G^2(\xi) + C_2 G(\xi) + C_3$ where C_1, C_2 and C_3 are arbitrary constants, which is generalized Riccati equation. They applied this method and obtained some solutions to the (3+1)-dimensional Jimbo-Miwa equation. Zhu (Zhu, 2008) revealed the generalized Riccati equation mapping with the extended tanh-function method and solved the (2+1)-dimensional Boiti-Leon-Pempinelle equation. Salas (Salas, 2008) expressed the projective Riccati equation method and solved the Caudrey-Dodd-Gibbon equation. Bekir and Cevikel (Bekir and Cevikel, 2011) attached the tanh-coth method combined with the Riccati equation. They found the traveling solutions to the (2 + 1)-dimensional breaking soliton equations by this method. Guo et al. (Guo et al., 2011) used the extended Riccati equation mapping method. They applied this method to obtain the traveling wave solutions for the diffusion reaction equation and the mKdV equation with variable coefficient. Zayed et al. (Zayed et al., 2014) applied the improved Riccati equation mapping method to constructed many exact solutions of a NLEE in nanobiosciences and biophysics which describe a model of microtubules as nonlinear RLC tranansmission line.

Wang et al. (Wang et al., 2008a) found a method, called the (G'/G)-expansion method which is straightforward and the algorithm for searching the unknown parameters is simple. In this method, it is assume that the solution of the NLEEs can be presented by a polynomial of (G'/G), i.e. $\sum_{i=0}^{n} a_i (G'/G)^i$, wherein $G = G(\xi)$ satisfies a second-order linear ordinary differential equation (ODE) $G'' + \lambda G' + \mu G = 0$, in which λ and μ are arbitrary constants. They applied this method and find some fresh solutions to the

Korteweg-de Vries equation, the mKdV equation, the variant Boussinesq equation and the Hirota-Satsuma equation. Wang et al. also (Wang et al., 2008b) used this method to obtain the traveling wave solutions involving parameters to the Broer-Kaup equation and the approximate long water wave equation. Later, several researchers have applied this method and solved different types of NLEEs. For example: Bekir (Bekir, 2008) applied the method and constructed traveling wave solutions to the Boussinesq equation, the modified Zakharov-Kuznetsov equation and the Konopeichenko-Dubrovsky equation. Zayed (Zayed, 2009a) found the solitary wave solutions involving parameters for some NLEEs in mathematical physics via the (2+1)-dimensional Painleve integrable Burgers equation, the (2+1)-dimensional Nizhnik-Novikov-Vesselov equation, the (2+1)-dimensional Boiti-Leon-Pempinelli equation and the (2+1)-dimensional dispersive long wave equations. Zayed and Gepreel (Zayed and Gepreel, 2009) also solved the combined Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the reaction-diffusion equation, the compound KdV-Burgers equation, and the generalized shallow water wave equation and they have found some traveling wave solutions of these equations. Zhang (Zhang, 2009) explored new application of this method to some special NLEEs, namely the Eckhaus equation, the modified Burgers equation and a kind of nonlinear reaction-diffusion equation, the balance numbers of which are not positive integers and obtained exact traveling wave solutions. Abazari (Abazari, 2010) investigated general exact solutions of three NLEEs namely, the Tzitzeica equation, the Dodd-Bullough-Mikhailov equation and the Tzitzeica-Dodd-Bullough equation. Taghizade and Neirameh (Taghizade and Neirameh, 2010) investigated the traveling wave solutions of the TRLW equation and the Gardner equation. Zayed (Zayed, 2010) found the solitary solutions of NLEEs in the mathematical physics via the (3+1)-dimensional potential- YTSF equation, the (3+1)-dimensional generalized shallow water

equation, the (3+1)- dimensional Kadomtsev-Petviashvili equation, the (3+1)-dimensional modified KdV Zakharov-Kuznetsev equation and the (3+1)-dimensional Jimbo-Miwa equation. Feng et al. (Feng et al., 2011) examined traveling wave solutions to the Kolmogorov-Petrovskii-Piskunov equation. Kheiri et al. (Kheiri et al., 2011) found some traveling wave solutions to the Burgers, Burgers-Huxley and modified Burgers-KdV equations. Naher et al. (Naher et al., 2011b) obtained some solitary solutions to the Caudrey-Dodd-Gibbon equation. Roozi and Mahmeiani (Roozi and Mahmeiani, 2011) searched some solitary wave solutions to the (2+1)-dimensional Kadomtsev-Petviashvili equation. Akbar et al. (Akbar et al., 2012a) obtained exact traveling wave solutions to the generalized Bretherton equation. Recently, Taha et al. (Taha et al., 2013) applied this method to NLEEs in plasma physics and found some fresh solitary wave solutions to the Schamel modified equation for ion-acoustic waves, one dimensional form of the Schamel-KdV equation and modified Kadomtsev-Petviashvili equation. Khan and Akbar (Khan and Akbar, 2015c) applied this method to obtain some solitary wave solutions to the fifth-order KdV equation in mathematical physics and the strain wave equation in microstructured solids in the engineering field.

Later on many researchers developed, modified and improved the (G'/G)-expansion method and also established this method in subsequent:

Zayed (Zayed, 2009b) offered a new approach of the (G'/G)-expansion method in which $G(\xi)$ satisfies the auxiliary equation: $\{G'(\xi)\}^2 = C_1G^4(\xi) + C_2G^2(\xi) + C_3$, called Jacobi elliptical equation, when C_1, C_2 and C_3 are arbitrary constants. He applied this method to obtain exact solutions of the NLEEs in mathematical physics via the (3+1)-dimensional potential-YTSF equation, the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation, the (3+1)-dimensional Kadomtsev-Petviashvili equation and the (1+1)-dimensional KdV equation.

Guo and Zhou (Guo and Zhou, 2010) provided the extended (G'/G)-expansion method. They used this method and made some fresh solutions to the Whitham-Broer-Like equations and the coupled Hirota-Satsuma KdV equations. In this method, the solution of the NLEEs were presented in the form $a_0 + \sum_{i=1}^{n}\left\{a_i\left(\frac{G'}{G}\right)^i + b_i\left(\frac{G'}{G}\right)^{i-1}\sqrt{\sigma\left\{1+\frac{1}{\mu}\left(\frac{G'}{G}\right)\right\}}\right\}$, where a_0, a_i and b_i $(i = 1, 2, 3, \cdots, n)$ are constants to be determined later, $\sigma = \pm 1$ and $G(\xi)$ satisfies the linear ODE $G'' + \mu G = 0$, when μ is a arbitrary constant. Then many researchers applied this extended method and found the traveling wave solutions to some NLEEs, such as: Zayed and Al-Joudi (Zayed and Al-Joudi, 2010) solved the (1+1)-dimensional modified Benjamin-Bona-Mahony (BBM) equation, the (2+1)-dimensional typical breaking soliton equation, the (1+1)-dimensional classical Boussinesq equations and the (2+1)-dimensional Broer-Kaup-Kuperschmidt equations in mathematical physics. Zayed and El-Malky (Zayed and El-Malky, 2011) constructed the traveling solutions to some (3+1)-dimensional NLEEs in mathematical physics, namely the nonlinear (3+1)-dimensional potential-YTSF equation and the (3+1)-dimensional generalized shallow water equation. Zhang et al. (Zhang et al., 2010) recruited the improved (G'/G)-expansion method, in which the solutions of NLEEs were presented in the form $\sum_{i=-n}^{n} a_i\left(\frac{G'}{G}\right)^i$ and they obtained the traveling wave solutions to the Zakharov-Kuznetsov-BBM equation and the (2+1)-dimensional dispersive long wave equations. Then some researchers used this method and illustrated the traveling wave solutions to NLEEs, such as: Zayed and Gepreel (Zayed and Gepreel, 2010) found the traveling wave solutions to the Konopelchenko-Dubrovsky equation, the Karsten-Krasil Shchik equation, the Whitham-Broer-Kaup equation and the fifth-order KdV equations. Naher and Abdullah (Naher and Abdullah, 2012a) obtained some traveling wave solutions of the combined KdV-mKdV equation. Naher and Abdullah (Naher and Abdullah, 2012b) also